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Note

Packing trees into complete bipartite graphs

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ABSTRACT

In 1976, Gyárfás and Lehel conjectured that any trees $T_2, \dots, T_n$ with 2 through n vertices pack into K_n, the complete graph on n vertices, that is, the trees $T_2, \dots, T_n$ appear as edge-disjoint subgraphs of K_n. This conjecture is still unresolved.

We examine an analogous conjecture for packing trees into complete bipartite graphs. Let $T_{a,a}$ denote a tree whose partite sets both have size a, which we call a *balanced* tree. We conjecture that any trees $T_{1,1}, \dots, T_{n,n}$ pack into $K_{n,n}$, the complete bipartite graph on $2n$ vertices.

We begin by establishing that if a and n are integers with $n \geq 3$ and $a < \lfloor \sqrt{7/18}n \rfloor$, then any balanced trees $T_{1,1}, \dots, T_{a,a}$ pack into $K_{n,n}$.

We also show that if any degree sequence for the first partite set is specified for each tree, then there exist balanced trees $T_{1,1}, \dots, T_{n,n}$ with these vertex degrees that pack into $K_{n,n}$.

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1. Introduction

The graphs $G_1, \dots, G_k$ *pack* into a graph H if $G_1, \dots, G_k$ appear as edge-disjoint subgraphs of H. In 1976, Gyárfás and Lehel [6] conjectured that any trees $T_2, \dots, T_n$ with 2 through n vertices pack into K_n, the complete graph on n vertices. This conjecture is still unresolved.

A number of partial results related to this conjecture have been shown. For example, Bollobás [1] showed that $T_2, \dots, T_r$ pack into K_n if $r \leq n/\sqrt{2}$. Later, Hobbs et al. [2] showed that any three trees T_n, T_{n-1}, T_{n-2} pack into K_n.

In addition, variations on the original conjecture have been examined, including packings into complete bipartite graphs: several such variations are summarized in [5]. In [2], Hobbs et al. conjectured that $T_2, \dots, T_n$ pack into the complete bipartite graph $K_{n-1,\lceil n/2 \rceil}$.

For positive integers a and b, let $T_{a,b}$ denote a tree with a bipartition into sets of sizes a and b. If $a = b$, then the sets of the bipartition have equal size; in this case we say that the tree is *balanced*.

Observe that $T_{a,a}$ has $2a-1$ edges. The observation that $\sum_{a=1}^{n}(2a-1) = n^2$, exactly the number of edges in the complete bipartite graph $K_{n,n}$, leads to the following conjecture, an analogue for bipartite graphs of the Gyárfás–Lehel conjecture.

Conjecture 1.1. *Any balanced trees $T_{1,1}, \dots, T_{n,n}$ pack into $K_{n,n}$.*

For small trees, the conjecture is straightforward: up to isomorphism, there are only three balanced trees on six vertices and eight balanced trees on eight vertices. It can be quickly verified that Conjecture 1.1 holds for $n \leq 4$.

In Section 2 we prove the bipartite analogue of the result of Bollobás [1] concerning lists of small trees.

Theorem 1.2. *If a and n are integers with $n \geq 3$ and $a < \lfloor \sqrt{7/18}n \rfloor$, then any balanced trees $T_{1,1}, \dots, T_{a,a}$ pack into $K_{n,n}$.*

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In Section 3 we prove that if any degree sequence for the first partite set is specified for each tree, then there exist balanced trees $T_{1,1}, \ldots, T_{n,n}$ with these vertex degrees that pack into $K_{n,n}$.

Theorem 1.3. *Fix $n \in \mathbb{N}$. Given, for each $1 \le k \le n$, positive integers $a_1^k, \ldots, a_k^k$ with $\sum_{i=1}^k a_i^k = 2k-1$, there exist balanced trees $T_{1,1}, \ldots, T_{n,n}$ such that*

1. *for each k, the vertices in the first partite set of $T_{k,k}$ have degrees $a_1^k, \ldots, a_k^k$;*
2. *the trees $T_{1,1}, \ldots, T_{n,n}$ pack into $K_{n,n}$.*

2. A lemma of Yuster and packing largest and smallest trees

We recall the following lemma due to Yuster [7].

Lemma 2.1 (*Yuster*)**.** *Let H be a bipartite graph with partite sets H_1 and H_2 of sizes h_1 and h_2, respectively, with $h_1 \le h_2$. Let T be a tree whose partite sets have sizes k_1 and k_2. If $k_1 \le h_1$, $k_2 \le h_2$ and $|E(H)| \ge k_2h_1 + k_1h_2 + k_1 + k_2 - h_1 - h_2 - k_1k_2$, then H contains a subgraph isomorphic to T.*

In Yuster's proof of this result, the subgraph isomorphic to T has k_1 vertices in the partite set of size h_1 and k_2 vertices in the partite set of size h_2.

Applying Yuster's lemma in the case where T is a balanced tree on $2k$ vertices, the restriction becomes

$$|E(H)| \ge (h_1 + h_2)(k-1) + 2k - k^2.$$

The following is an immediate consequence.

Corollary 2.2. *Let H be a subgraph of $K_{n,n}$, and let $k \le n$. If $|E(H)| \ge 2n(k-1) + 2k - k^2$, then H contains every balanced tree on 2k vertices.*

This has consequences for packings of balanced trees, as we shall see in the next section.

We are interested in packing balanced trees $T_{1,1}, \ldots, T_{n,n}$ into $K_{n,n}$. We first consider what happens if we just start at the end of the list and start packing trees. Observe that any two balanced trees $T_{n,n}$ and $T_{n-1,n-1}$ pack into $K_{n,n}$: in the biadjacency matrix of $K_{n,n}$, the edges of the tree $T_{n,n}$ can be placed on or below the main diagonal, and the edges of $T_{n-1,n-1}$ can be placed above the main diagonal.

If we hope to pack in more trees, then we cannot arbitrarily place $T_{n,n}$ in the lower triangle and $T_{n-1,n-1}$ in the upper triangle. Consider the *double-star*, the balanced tree with $2n$ vertices having two vertices of degree n and $2n-2$ vertices of degree 1:

$$\begin{bmatrix} 5 & 4 & \cdot & \cdot & 4 \\ \cdot & 5 & 4 & \cdot & 4 \\ \cdot & \cdot & 5 & 4 & 4 \\ 5 & 5 & \cdot & 5 & 4 \\ \cdot & 5 & 5 & \cdot & 5 \end{bmatrix}.$$

Here we have two balanced trees, one with 10 vertices and the other with 8, packed into $K_{5,5}$ in such a way that the double-star on six vertices will not fit. The double-star requires an open position for its central edge that has 2 other open positions in its row and in its column.

Not surprisingly, we can achieve somewhat more success by beginning with the smallest trees, in a manner modeled after Yuster's results for non-balanced trees [7].

Theorem 1.2. *If a and n are integers with $n \ge 3$ and $a < \lfloor \sqrt{7/18}n \rfloor$, then any balanced trees $T_{1,1}, \ldots, T_{a,a}$ pack into $K_{n,n}$.*

Proof. Certainly $K_{n,n}$ contains a copy of the largest tree $T_{a,a}$. Assume that $T_{a,a}, T_{a-1,a-1}, \ldots, T_{k+1,k+1}$ have already been packed into $K_{n,n}$ for some k with $1 < k < a$. Let H be the spanning subgraph of $K_{n,n}$ that contains all the edges not yet used in the packing. We have

$$\begin{aligned} |E(H)| &= n^2 - ((2a-1) + (2a-3) + \cdots + (2k-1)) \\ &= n^2 - a^2 + (k-1)^2. \end{aligned}$$

By Corollary 2.2, H contains the next tree $T_{k,k}$ if $n^2 - a^2 + (k-1)^2 > 2k - 2n + 2kn - k^2$; that is, if $n^2 - a^2 + 2k^2 - 4k + 2n - 2kn + 1 > 0$. The function $f(k) = n^2 - a^2 + 2k^2 - 4k + 2n - 2kn + 1$ is minimized when $2k = 2 + n$, at which point $f(k) = n^2 - 2a^2 - 2$. Now, if $a < \lfloor \sqrt{7/18}n \rfloor$, then

$$n^2 - 2a^2 - 2 > n^2 - 2(7/18)n^2 - 2 = (2/9)n^2 - 2.$$

Since $n \ge 3$, we have $(2/9)n^2 - 2 \ge 0$, so $f(k) > 0$, as required. □

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