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Overlarge sets of resolvable idempotent quasigroups☆

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ABSTRACT

An idempotent quasigroup $(X, \circ)$ of order v is called resolvable (denoted by RIQ(v)) if the set of $v(v-1)$ non-idempotent 3-vectors $\{(a, b, a \circ b) : a, b \in X, a \neq b\}$ can be partitioned into $v-1$ disjoint transversals. An overlarge set of idempotent quasigroups of order v, briefly by OLIQ(v), is a collection of $v+1$ IQ(v)s, with all the non-idempotent 3-vectors partitioning all those on a $(v+1)$-set. An OLRIQ(v) is an OLIQ(v) with each member IQ(v) being resolvable. In this paper, it is established that there exists an OLRIQ(v) for any positive integer $v \geq 3$, except for $v = 6$, and except possibly for $v \in \{10, 11, 14, 18, 19, 23, 26, 30, 51\}$. An OLIQ$^{\diamond}(v)$ is another type of restricted OLIQ(v) in which each member IQ(v) has an idempotent orthogonal mate. It is shown that an OLIQ$^{\diamond}(v)$ exists for any positive integer $v \geq 4$, except for $v = 6$, and except possibly for $v \in \{14, 15, 19, 23, 26, 27, 30\}$.

1. Introduction

Let X be a v-set on which a binary operation $\circ$ is defined. The pair $(X, \circ)$ is a *quasigroup of order* v if, for any two elements, $a, b \in X$, the two equations $a \circ x = b$ and $x \circ a = b$ have exactly one solution x each in X. A quasigroup $(X, \circ)$ is *idempotent*, denoted by IQ(v), if it satisfies $a \circ a = a$ for any $a \in X$.

Let P be a set of ordered pairs of X. P is called a *transversal* of a quasigroup $(X, \circ)$ if $\{a : (a, b) \in P\} = \{b : (a, b) \in P\} = \{a \circ b : (a, b) \in P\} = X$. We also refer the transversal to $T = \{(a, b, a \circ b) : (a, b) \in P\}$. Two quasigroups $(X, \circ)$ and $(X, \cdot)$ are *orthogonal* if $\{(a \circ b, a \cdot b) : a, b \in X\} = X \times X$. Obviously a quasigroup of order v has an orthogonal mate if and only if it has a resolution of v disjoint transversals. For an IQ(v), the diagonal cells form a transversal, which we call the *idempotent transversal*. If an IQ(v) has a resolution $\{T_0, T_1, \ldots, T_{v-1}\}$, where T_0 is the idempotent transversal, then we say that the IQ(v) is *resolvable* and denote it by RIQ(v). There are other possible resolutions for an IQ. For instance, an IQ(v) may have an idempotent orthogonal mate. To put it in other words, if the IQ(v) is defined on $X = \{x_0, x_1, \ldots, x_{v-1}\}$, then it has a resolution $\{T_0, T_1, \ldots, T_{v-1}\}$ such that $(x_i, x_i, x_i) \in T_i$, $0 \leq i \leq v-1$. For convenience we denote such an IQ(v) by IQ$^{\diamond}(v)$.

Let $(X, \circ)$ be an IQ(v) and $\mathcal{A} = \{(a, b, a \circ b) : a, b \in X, a \neq b\}$. Then we represent the IQ(v) as an *ordered design* of order v (OD(v)) $(X, \mathcal{A})$. That is, we obtain a $v(v-1) \times 3$ array such that (1) each row has 3 distinct elements of X, and (2) each two columns contains each ordered pair of distinct elements of X precisely once. Conversely, an OD(v) can be enlarged to an IQ(v) by supplementing v idempotent rows.

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Two OD(v)s defined on the same set are *disjoint* if they have no common 3-vectors. Let X be a set of $v+1$ elements. If all the $(v+1)v(v-1)$ 3-vectors of distinct elements of X can be partitioned into $v+1$ pairwise disjoint OD(v)s $(X \setminus \{x\}, \mathcal{A}_x)$, $x \in X$, then the collection of $\{(X \setminus \{x\}, \mathcal{A}_x) : x \in X\}$ is called an *overlarge set* of ordered designs of order v. By supplementing v idempotent rows to each OD(v) in the overlarge set, we obtain an overlarge set of idempotent quasigroups of order v, simply denoted by OLIQ(v). The problem of OLIQs was studied in [2,9,11] and an OLIQ(v) exists if and only if $v \geq 3$ and $v \neq 6$. With the complete determination of the existence of an OLIQ(v), it is worthwhile to consider OLIQs with some restricted conditions. An overlarge set of resolvable IQ(v)s, denoted by OLRIQ(v), is an OLIQ(v) with each member IQ(v) resolvable. On the other hand, if each member in an OLIQ(v) is an IQ$^\diamond$(v), then we denote the OLIQ(v) by OLIQ$^\diamond$(v).

Let X be a v-set. A *pairwise balanced design* (PBD) of order v is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a family of subsets of X (called *blocks*) such that each unordered pair of X is contained in exactly one block of $\mathcal{B}$. A PBD(v, K) denotes a PBD of order v with block sizes from the set K. A PBD($v, \{3\}$) is a *Steiner triple system* of order v, denoted by STS(v). Let $(X, \mathcal{B})$ be an STS(v). If there exists a partition $\{P_1, P_2, \ldots, P_{(v-1)/2}\}$ of $\mathcal{B}$ such that each part P_i forms a *parallel class*, i.e., a partition of X, then the STS(v) is resolvable. A resolvable STS(v) is usually called a *Kirkman triple system* of order v (briefly KTS(v)). It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ (see [8]).

A *large set* of STS(v)s, denoted by LSTS(v), is a partition of all triples on v points into $v-2$ disjoint STS(v)s. An overlarge set of STS(v)s, denoted by OLSTS(v), is a partition of all triples on $v+1$ points into $v+1$ disjoint STS(v)s. A large (or overlarge) set of Kirkman triple systems of order v, denoted by LKTS(v) (or OLKTS(v)), is an LSTS(v) (or OLSTS(v)) where each member STS(v) is a KTS(v). An OLSTS(v) exists for any $v \equiv 1, 3 \pmod 6$ as the collection of $v+1$ derived designs of a Steiner quadruple system of order $v+1$ (for more details, see [4]). The existence problem of LSTSs was also completely solved, mainly by Lu [6,7] but also by Teirlinck [10]. However, the research on LKTSs and OLKTSs progressed at a slow pace; see for instance [12,13] for the latest progress. Subsequently, some research focuses on a few generalized analogues to LKTSs and OLKTSs; see [5] for large sets and overlarge sets of resolvable (oriented) triple systems, [14,16] for large sets of resolvable idempotent quasigroups. In this paper we investigate OLRIQs and display an almost solution to the existence problem of OLRIQs, for which there is little literature to the best of our knowledge. But observe that an OLKTS(v) implies an OLRIQ(v), which can be used to produce some preliminary results on OLRIQs. OLIQ$^\diamond$s were investigated in [15], as an example of P3BD-closed sets, and then shown to exist for $v \geq 4$, $v \neq 6$, with a handful of possible exceptions. We also improve this result in the present paper. We only record the following result for later use.

Lemma 1.1 ([5,12]). *There exists an OLKTS(v) and hence an OLRIQ(v) for $v \in \{3, 9, 15, 27\}$.*

2. A PBD construction

This section exhibits a PBD construction for OLRIQs.

We will use a restricted OLIQ with each member being an RIQ or IQ$^\diamond$. Let $|B| = k$, $\infty \notin B$, and $\{((B \cup \{\infty\}) \setminus \{x\}, \mathcal{A}_x) : x \in B \cup \{\infty\}\}$ be an OLIQ(k). If each $\mathcal{A}_x$, $x \in B$, is an RIQ(k) and $\mathcal{A}_\infty$ is an IQ$^\diamond$(k), then we denote the OLIQ by OLIQ†(k).

Construction 2.1. Let $(X, \mathcal{B})$ be a PBD(v, K) with $2, 3, 6 \notin K$ and $\infty \notin X$. For any $B \in \mathcal{B}$, if there exists $x_0 \in X$ such that, over $B \cup \{\infty\}$, an OLRIQ($|B|$) exists if $x_0 \in B$ and an OLIQ†($|B|$) exists if $x_0 \notin B$, then there exists an OLRIQ(v) over $X \cup \{\infty\}$.

Proof. In the assumed PBD(v, K)$(X, \mathcal{B})$, let ab denote the unique block of $\mathcal{B}$ containing both a and b, where a and b are two distinct elements of X. For any $B \in \mathcal{B}$, let $\{((B \cup \{\infty\}) \setminus \{x\}, \mathcal{B}_B^x) : x \in B \cup \{\infty\}\}$ be an OLRIQ($|B|$) if $x_0 \in B$, or an OLIQ†($|B|$) if $x_0 \notin B$. So each IQ($|B|$) $= ((B \cup \{\infty\}) \setminus \{x\}, \mathcal{B}_B^x)$ has an orthogonal mate $((B \cup \{\infty\}) \setminus \{x\}, \mathcal{A}_B^x)$. Furthermore, for any $x \in B$ and $a \in (B \cup \{\infty\}) \setminus \{x\}$, $(a, a, \infty) \in \mathcal{A}_B^x$; and for any $a \in B$, $(a, a, x_0) \in \mathcal{A}_B^\infty$ if $x_0 \in B$, or $(a, a, a) \in \mathcal{A}_B^\infty$ if $x_0 \notin B$.

For any $x \in X$, define

$$\mathcal{B}_x = \left(\bigcup_{x \in B, B \in \mathcal{B}} (\mathcal{B}_B^x \setminus \{(\infty, \infty, \infty)\}) \right) \bigcup \{(\infty, \infty, \infty)\},$$

$$\mathcal{B}'_x = \left(\bigcup_{x \in B, B \in \mathcal{B}} (\mathcal{A}_B^x \setminus \{(\infty, \infty, \infty)\}) \right) \bigcup \{(\infty, \infty, \infty)\}.$$

For any $B \in \mathcal{B}$, since $|B| \geq 4$ and $|B| \neq 6$, there is a pair of orthogonal IQ($|B|$)s $(B, \circ_B)$ and $(B, \cdot_B)$. Define for $x \in X$

$$\mathcal{C}_x = \{(a, b, c) : a \circ_{ab} b = c \circ_{cx} x, \{a, b, c\} \not\subseteq B \text{ for any } B \in \mathcal{B}\},$$
$$\mathcal{C}'_x = \{(a, b, c) : a \cdot_{ab} b = c \cdot_{cx} x, \{a, b, c\} \not\subseteq B \text{ for any } B \in \mathcal{B}\},$$
$$\mathcal{D}_x = \mathcal{B}_x \cup \mathcal{C}_x, \qquad \mathcal{D}'_x = \mathcal{B}'_x \cup \mathcal{C}'_x.$$