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Strong edge-coloring for cubic Halin graphs

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ABSTRACT

A strong edge-coloring of a graph G is a function that assigns to each edge a color such that two edges within distance two apart must receive different colors. The minimum number of colors used in a strong edge-coloring is the *strong chromatic index* of G. Lih and Liu (2011) [14] proved that the strong chromatic index of a cubic Halin graph, other than two special graphs, is 6 or 7. It remains an open problem to determine which of such graphs have strong chromatic index 6. Our article is devoted to this open problem. In particular, we disprove a conjecture of Shiu et al. (2006) [18] that the strong chromatic index of a cubic Halin graph with characteristic tree a caterpillar of odd leaves is 6.

1. Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges e and e' in a graph is the minimum k for which there is a sequence $e_0, e_1, \ldots, e_k$ of distinct edges such that $e = e_0$, $e' = e_k$, and e_{i-1} shares an end vertex with e_i for $1 \le i \le k$. A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges within distance two apart must receive different colors. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. The *strong chromatic index* of a graph G, denoted by $\chi_s'(G)$, is the minimum k such that G admits a strong k-edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [8,9] for cubic planar graphs. A trivial upper bound is that $\chi_s'(G) \le 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ. Fouquet and Jolivet [8] established a Brooks type upper bound $\chi_s'(G) \le 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [19]. The following conjecture was posed by Erdős and Nešetřil [5,6] and revised by Faudree et al. [7].

Conjecture 1. *For any graph G of maximum degree Δ,*

$$\chi_s'(G) \le \begin{cases} \dfrac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \dfrac{5}{4}\Delta^2 - \dfrac{1}{2}\Delta + \dfrac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Faudree et al. [7] also asked whether $\chi_s'(G) \le 9$ if G is cubic planar. If this upper bound is proved to be true, it would be best possible. For graphs with maximum degree $\Delta = 3$, Conjecture 1 was verified by Andersen [1] and by Horák et al. [12]

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independently. For $\Delta = 4$, while Conjecture 1 says that $\chi_s'(G) \le 20$, Horák [11] obtained $\chi_s'(G) \le 23$ and Cranston [4] proved $\chi_s'(G) \le 22$. In addition to Conjecture 1, other aspects of strong edge-coloring have also been studied extensively (cf. [1–3,10,13–18,20]).

The main theme of this paper is to study strong edge-coloring for the following planar graphs. A *Halin graph* $G = T \cup C$ is a plane graph consisting of a plane embedding of a tree T each of whose interior vertex has degree at least 3, and a cycle C connecting the leaves (vertices of degree 1) of T such that C is the boundary of the exterior face. The tree T and the cycle C are called the *characteristic tree* and the *adjoint cycle* of G, respectively. Strong chromatic index for Halin graphs was first considered by Shiu et al. [18] and then studied in [19,13,14].

A *caterpillar* is a tree whose removal of leaves results in a path called the *spine* of the caterpillar. For $k \ge 1$, let $\mathcal{G}_k$ be the set of all cubic Halin graphs whose characteristic trees are caterpillars with $k + 2$ leaves. For a graph $G = T \cup C$ in $\mathcal{G}_k$, let P: $v_1, v_2, \ldots, v_k$ be the spine of T and each v_i is adjacent to a leaf u_i for $1 \le i \le k$ with v_1 (resp. v_k) adjacent to one more leaf $u_0 = v_0$ (resp. $u_{k+1} = v_{k+1}$). We draw G on the plane by putting the path v_0Pv_{k+1} horizontally in the middle, and the pending edges (leaf edges) v_iu_i, $1 \le i \le k$, by either up or down edges vertically to P. See Fig. 1 for an example of $\mathcal{G}_8$.

From this drawing, we associate G with a list of positive integers $(n_1, n_2, \ldots, n_r)$, where n_i is the number of maximum consecutive up or down edges, starting from the leftmost to the rightmost on P. We use $G_{n_1,n_2,\ldots,n_r}$ to denote this graph. For instance, the graph in Fig. 1 is $G_{2,3,3}$. Notice that $n_1 + n_2 + \cdots + n_r = k$. For a special case when these pending edges are all in the same direction (up or down), the graph G_k is called the *necklace* and is denoted by Ne_k in [18]. Notice that G_k is the only graph in $\mathcal{G}_k$ for $k \le 3$.

Observation 1. $G_{n_1,n_2,\ldots,n_r} \cong G_{n_r,\ldots,n_2,n_1}$.

Observation 2. $G_{n_1,n_2,\ldots,n_r,1} \cong G_{n_1,n_2,\ldots,n_r+1}$.

It is easy to see that $\chi_s'(G) \ge 6$ for any $G \in \mathcal{G}_k$, $k \ge 1$. Shiu et al. [18] obtained the following results.

- $$\chi_s'(G_k) = \begin{cases} 9, & k = 2; \\ 8, & k = 4; \\ 7, & k \text{ is even and } k \ge 6; \\ 6, & k \text{ is odd.} \end{cases}$$
- If $G \in \mathcal{G}_k$ with $k \ge 4$, then $6 \le \chi_s'(G) \le 8$.
- If G is a cubic Halin graph, then $6 \le \chi_s'(G) \le 9$.

Moreover, the authors [18] raised the following conjectures.

Conjecture 2. *If $G \in \mathcal{G}_k$ with $k \ge 5$, then $\chi_s'(G) \le 7$.*

Conjecture 3. *If $G \in \mathcal{G}_k$ with odd $k \ge 5$, then $\chi_s'(G) = 6$.*

Conjecture 4. *If $G = T \cup C$ is a Halin graph, then $\chi_s'(G) \le \chi_s'(T) + 4$.*

Faudree et al. [7] proved, for any tree T, it holds that $\chi_s'(T) = \max_{uv \in E(T)}(\deg(u) + \deg(v) - 1)$. Conjecture 4 was confirmed by Lai et al. [13], who proved a stronger result that $\chi_s'(G) \le \chi_s'(T) + 3$ for any Halin graph $G = T \cup C$ other than G_2 and wheels W_n with $n \not\equiv 0 \pmod 3$, where $W_n = K_{1,n} \cup C_n$. Note that $\chi_s'(W_5) = \chi_s'(K_{1,5}) + 5$, and $\chi_s'(G) = \chi_s'(T) + 4$ for $G = G_2$ or $G = W_n$ with $n \not\equiv 0 \pmod 3$ and $n \ne 5$.

Conjecture 2 was confirmed by Lih and Liu [14], who proved a more general result that $\chi_s'(G) \le 7$ is true for any cubic Halin graph other than G_2 and G_4. Hence, strong chromatic index for any cubic Halin graph $G \ne G_2, G_4$ is either 6 or 7.

It remains open to determine the cubic Halin graphs G with $\chi_s'(G) = 6$ (or the ones with $\chi_s'(G) = 7$). Our aim is to investigate this problem. In particular, we establish methods that can be used to study the graphs $\mathcal{G}_k$. As a result, we discover counterexamples to Conjecture 3. We prove that for any $k \ge 7$, there exists graph $G \in \mathcal{G}_k$ with $\chi_s'(G) = 7$, and for any $k \ne 2, 4$, there exists $G \in \mathcal{G}_k$ (other than necklaces) with $\chi_s'(G) = 6$. In Section 4, we determine the value of $\chi_s'(G)$ for some special families of graphs G in $\mathcal{G}_k$.

2. Cubic Halin graphs G with $\chi_s'(G) = 6$

This section gives some cubic Halin graphs with strong chromatic index 6. We begin with the development of several general transformation theorems for Halin graphs.

For a positive integer r, an *r-tail* of a tree T is a path P_r: $v_1, v_2, \ldots, v_r, v_{r+1}$ in which v_1 is not a leaf but all vertices in $L_i = \{u \notin P : uv_i \in E(T)\}$ are leaves for $1 \le i \le r$. For integer $s < r$, *cutting* P_s from T means deleting the vertices $\{v_1, v_2, \ldots, v_{s-1}\} \cup_{1 \le i \le s} L_i$ from T, which results in a tree denoted by $T \ominus P_s$. Notice that v_s becomes a leaf adjacent to v_{s+1} in $T \ominus P_s$.