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On small subgraphs in a random intersection digraph

Valentas Kurauskas *

Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania

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ABSTRACT

Given a set of vertices V and a set of attributes W let each vertex $v \in V$ include an attribute $w \in W$ into a set $S^-(v)$ with probability p_- and let it include w into a set $S^+(v)$ with probability p_+ independently for each $w \in W$. The random binomial intersection digraph on the vertex set V is defined as follows: for each $u, v \in V$ the arc uv is present if $S^-(u)$ and $S^+(v)$ are not disjoint. For any $h = 2, 3, \ldots$ we determine the birth threshold of the complete digraph on h vertices and describe the configurations of intersecting sets that realise the threshold.

1. Introduction

In the random intersection graph introduced by Karoński, Scheinerman and Singer-Cohen [13] (see also [7] and Godehardt and Jaworski [10]) we have a set of vertices V of size n and an additional set W of attributes (also sometimes called properties or keys) of size m. Each vertex v of V chooses a random subset of attributes in W and an edge uv is added to the graph if and only if the subsets of u and v intersect.

We consider a directed random intersection graph D on the vertex set V defined as follows (see [4]). Let each vertex v choose not one, but two random subsets: an "in"-subset, $S^+(v)$, and an "out"-subset, $S^-(v)$. An arc from vertex u to v is inserted in D whenever $S^-(u)$ intersects $S^+(v)$. Assuming, in addition, that each attribute $w \in W$ is included in the subset $S^-(v)$ with probability p_- and in the subset $S^+(v)$ with probability p_+ independently and independently of all other inclusions, we obtain a *random binomial intersection digraph* denoted $D = D(n, m, p_-, p_+)$.

In [18] a network of co-authors of mathematical papers is mentioned as an illustration for random intersection graphs. One might alternatively define a *citation digraph* where V is a set of mathematicians and we draw an arc from u to v if and only if u has cited v. The underlying set W here would be the set of all mathematical papers; and $S^-(u)$ (respectively, $S^+(u)$) would correspond to the set of papers u has cited (respectively, co-authored).

The random intersection graph model has received a lot of attention recently due to several different applications [8,10,13]. Properties such as thresholds for small graphs [13], degree distribution [3,6,18], formation of the giant component [1,5], connectivity [2,15] and clustering [6] have been studied; see also [17,9,14]. In some applications considering directed intersection graphs makes sense and might lead to more precise/adequate models. In particular, one may obtain a digraph with power law indegree distribution and bounded outdegree distribution. In addition these digraphs have a clustering property when m is of order n [4].

In the problem of determining the birth threshold of small subgraphs one is interested in the question of how dense a graph should be to have a desired subgraph with certainty. There is a rich literature devoted to birth thresholds in random

* Correspondence to: Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania.
E-mail address: valentas@gmail.com.

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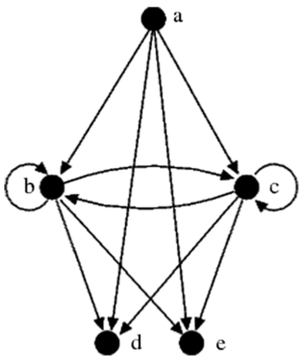

Fig. 1. The diclique $(\{a, b, c\}; \{b, c, d, e\})$.

graphs with independent edges where each edge appears with the same probability; see, e.g., Chapter 3 of [12]. The threshold for a random (binomial) intersection graph to contain a fixed subgraph has been studied in [13].

Here we consider a similar problem for random intersection digraphs. Let $\overrightarrow{K}_h$ be the complete digraph on vertex set $[h] = \{1, \ldots, h\}$ containing arcs xy and yx for each pair of distinct vertices $x, y \in [h]$. We aim to determine critical values of the parameters for $D(n, m, p_-, p_+)$ to have with a high probability a subgraph isomorphic to $\overrightarrow{K}_h$.

Given two finite sets C^- and C^+ we consider the ordered pair $C = (C^-; C^+)$ and the digraph $D(C)$ on the vertex set $C^- \cup C^+$ with the set of arcs $\{uv : u \in C^-, v \in C^+\}$. We call the pair C a *diclique*. We say that C is *proper* if C^-, C^+ are non-empty, otherwise say that it is *improper*. We remark that if the digraph $D(C)$ is non-empty then C must be proper and $D(C) \neq D(C')$ for $C \neq C'$. Therefore we will identify a proper diclique C with the corresponding digraph $D(C)$; see Fig. 1.

To our knowledge, the diclique digraphs were first studied by Haralick [11], but in a different context.

In the random digraph D with vertex set V and attribute set W each attribute $w \in W$ defines a diclique $C(w) = (C^-(w); C^+(w))$ given by $C^-(w) = \{v \in V : w \in S^-(v)\}$ and $C^+(w) = \{v \in V : w \in S^+(v)\}$. It is convenient to interpret each attribute $w \in W$ as a distinct colour. Then all the attributes in W give rise to a family of dicliques of different colours which covers all arcs of D.

The paper is organised as follows. In the next section we present our main results. In Section 3 we give a general lemma for the birth threshold of a fixed directed graph H. In Section 4 we study a few special diclique covers of $\overrightarrow{K}_h$ and prove our main results Theorems 2.1 and 2.2.

We remind some standard notation used in the paper. For functions $f, g : \mathbb{N} \to \mathbb{R}_+$ we write $f \sim g$ if $\lim_{k\to\infty} f(k)/g(k) = 1$. We write $f = O(g)$ if $\limsup_{k\to\infty} f(k)/g(k) < \infty$, $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$. We write $f = o(g)$ if $f(k)/g(k) \to 0$.

Finally, thanks to an anonymous reviewer the author became aware of a related and very relevant result on the Poisson approximation of the number of cliques in sparse random intersection graphs by Rybarczyk and Stark [16].

2. Results

Before stating our main results we need to introduce some definitions related to dicliques. Without loss of generality we will assume that the set of vertices of the random digraph D is $V = [n]$.

For any diclique C, we call $V(C) = C^- \cup C^+$ the vertex set of C. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$ be a family of dicliques, (we allow $\mathcal{C}$ to be a multiset and in this paper we consider only finite families $\mathcal{C}$). Let us denote by $V(\mathcal{C})$ the union of all vertices of the dicliques, $V(\mathcal{C}) = \bigcup V(C_i)$. We say that D *contains* $\mathcal{C}$ if there are distinct attributes $w_1, \ldots, w_s \in W$, such that $C_i \subseteq C(w_i)$ for each $i = 1, \ldots, s$ (the set operations for dicliques are defined componentwise). Also, let us call a diclique family *proper* if all its dicliques are proper.

Let $\mathcal{C}$ be any diclique family with $V(\mathcal{C}) = \{v_1, \ldots, v_r\} \subseteq [n]$ and assume that $v_1 < \cdots < v_r$. For any set $S = \{x_1, \ldots, x_r\} \subseteq [n]$ with $x_1 < \cdots < x_r$, let us denote by $M(\mathcal{C}, S)$ the diclique family which is an image of $\mathcal{C}$ obtained by renaming v_i to x_i for each $i = 1, \ldots, r$. We call $M(\mathcal{C}, S)$ a *copy* of $\mathcal{C}$.

Each family of dicliques $\mathcal{C}$ defines a digraph $H = H(\mathcal{C})$ with vertex set $V(\mathcal{C})$: an arc is present in H whenever it is present in some $D(C)$, $C \in \mathcal{C}$. We say that the family $\mathcal{C}$ is a *diclique cover* of H.

The digraph $\overrightarrow{K}_h$ can be covered by dicliques in many different ways. Consider the following important symmetric diclique covers of $\overrightarrow{K}_h$:

- $\mathcal{C}_M = \{([h]; [h])\}$, the monochromatic diclique cover;
- $\mathcal{C}_R = E(\overrightarrow{K}_h)$, the rainbow diclique cover, where $E(\overrightarrow{K}_h)$ is the set of arcs of $\overrightarrow{K}_h$ and we identify each arc uv with the diclique $(\{u\}; \{v\})$;