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Extremal digraphs whose walks with the same initial and terminal vertices have distinct lengths

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ABSTRACT

Let D be a digraph of order n in which any two walks with the same initial vertex and the same terminal vertex have distinct lengths. We prove that D has at most $(n+1)^2/4$ arcs if n is odd and $n(n+2)/4$ arcs if n is even. The digraphs attaining this maximum size are determined.

1. Introduction and main results

Digraphs in this paper allow loops but do not allow multiple arcs unless otherwise stated. We follow the terminology in [1,2]. The number of vertices in a digraph is called its *order* and the number of arcs its *size*. For digraphs, cycles and walks will mean directed cycles and directed walks respectively.

For a given positive integer n, let $\Theta(n)$ denote the set of digraphs of order n in which any two walks with the same initial vertex and the same terminal vertex have distinct lengths. Thus, for a digraph D on the vertices $1, 2, \dots, n$, $D \in \Theta(n)$ if and only if for every pair of vertices i, j and for every positive integer k there is at most one walk of length k from i to j. Let $\theta(n)$ denote the maximum size of a digraph in $\Theta(n)$.

We consider the following problem.

Problem 1. For a given positive integer n, determine $\theta(n)$ and determine the digraphs in $\Theta(n)$ that attain the size $\theta(n)$.

The motivation for studying Problem 1 is to explore the relation between the size and the walks of a digraph. Intuitively $\theta(n)$ cannot be very large compared with n^2, while the structure of the extremal digraphs attaining $\theta(n)$ seems unclear. Recall that a square upper triangular matrix is called *strict* if its diagonal entries are zero. Throughout we denote by $J_{r,t}$ the $r \times t$ matrix with each entry equal to 1 and abbreviate $J_{t,t}$ as J_t. Our solution to Problem 1 is contained in the following main result.

Theorem 1. *Let n be a positive integer. Then*

$$\theta(n) = \begin{cases} \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd,} \\ \dfrac{n(n+2)}{4} & \text{if } n \text{ is even.} \end{cases}$$

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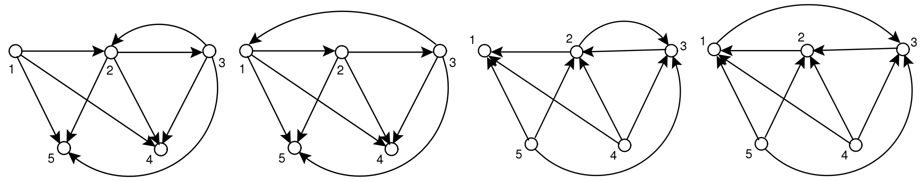

Fig. 1. The extremal loopless digraphs of order 5.

A digraph $D \in \Theta(n)$ has size $\theta(n)$ if and only if the adjacency matrix of D is permutation similar to

$$\begin{pmatrix} U & E & J_{r,t} \\ 0 & P & J_{s,t} \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

or its transpose, where P is a permutation matrix and it does appear, U is a strictly upper triangular matrix, there is exactly one entry 1 in each row of $(U\ E)$, $t = (n-1)/2$ if n is odd and $t = n/2 - 1$ or $n/2$ if n is even.

The corresponding theorem on loopless digraphs follows from Theorem 1 immediately: For $n \geq 2$ the maximum size remains $\theta(n)$ and the extremal digraphs attaining $\theta(n)$ are those whose adjacency matrices are permutation similar to the matrix in (1) or its transpose with the additional condition that P has zero diagonal entries. The four extremal loopless digraphs of order 5 are shown in Fig. 1.

Denote by $M_{m,n}\{0, 1\}$ the set of $m \times n$ 0–1 matrices, and abbreviate $M_{n,n}\{0, 1\}$ as $M_n\{0, 1\}$. For a given positive integer n, denote

$$\Gamma(n) = \{A \in M_n\{0, 1\} | A^k \in M_n\{0, 1\} \text{for every positive integer } k\}$$

and denote by $f(A)$ the number of 1's in a matrix A. Define $\gamma(n) = \max\{f(A) | A \in \Gamma(n)\}$.

It is well known [2, p. 72] that for $A \in M_n\{0, 1\}$ and a given positive integer k, $A^k \in M_n\{0, 1\}$ if and only if for every pair of vertices i, j (not necessarily distinct) there is at most one walk of length k from i to j in the digraph of A. Thus, considering the adjacency matrix of a digraph we see that Problem 1 is equivalent to the following

Problem 2. For a given positive integer n, determine $\gamma(n)$ and determine the matrices in $\Gamma(n)$ that attain $\gamma(n)$.

The solution to Problem 2 is the following equivalent matrix version of Theorem 1:

Theorem 2. *Let n be a positive integer. Then*

$$\gamma(n) = \begin{cases} \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd}, \\ \dfrac{n(n+2)}{4} & \text{if } n \text{ is even}. \end{cases}$$

For a matrix $A \in \Gamma(n)$, $f(A) = \gamma(n)$ if and only if A is permutation similar to

$$\begin{pmatrix} U & E & J_{r,t} \\ 0 & P & J_{s,t} \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

or its transpose, where P is a permutation matrix and it does appear, U is a strictly upper triangular matrix, there is exactly one entry 1 in each row of $(U\ E)$, $t = (n-1)/2$ if n is odd and $t = n/2 - 1$ or $n/2$ if n is even.

We will prove Theorem 2 in Section 2. A related problem with a fixed length of walks is studied in [3,5].

2. Proof of Theorem 2

To prove Theorem 2 we need several lemmas. Denote the digraph of $A \in M_n\{0, 1\}$ by $D(A)$. Use the notation $i \xrightarrow[d_{ij}]{W_{ij}} j$ to indicate that the walk W_{ij} from i to j is of length d_{ij}. If there is a walk W_{ij} from i to j of length d_{ij} and a walk W_{jk} from j to k of length d_{jk}, we write $i \xrightarrow[d_{ij}]{W_{ij}} j \xrightarrow[d_{jk}]{W_{jk}} k$, and so on.

Lemma 3. *Let $D \in \Theta(n)$ with $n \geq 2$. Then D is strongly connected if and only if D is a cycle.*

Proof. If D is a cycle, then it is clearly strongly connected. Conversely suppose $D \in \Theta(n)$ is strongly connected. To prove that D is a cycle it suffices to show that the outdegree of each vertex of D is 1. Since D is strongly connected, the outdegree