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Balanced generic circuits without long paths

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ABSTRACT

We call a graph $G = (V, E)$ a (k, ℓ)-*circuit* if $|E| = k|V| - \ell + 1$ and every $X \subset V$ with $2 \leq |X| \leq |V| - 1$ induces at most $k|X| - \ell$ edges. A (2, 3)-circuit is also called a *generic circuit*. We say that a graph is *balanced* if the difference between its maximum and minimum degrees is at most 1. Graver et al. asked in Graver et al. (1993) [7] whether a balanced generic circuit always admits a decomposition into two disjoint Hamiltonian paths. We show that this does not hold, moreover there are balanced $(k, k+1)$-circuits for all $k \geq 2$ which do not contain any Hamiltonian path nor a path longer than $|V|^\lambda$ for $\lambda > \frac{\log 8}{\log 9} \simeq 0, 9464$.

1. Introduction

All graphs considered are undirected and simple (i.e., may not contain loops or multiple edges). Let $G = (V, E)$ be a graph, and k, ℓ nonnegative integers. Let $i(X)$ denote the number of edges induced by a subset X of V. G is called (k, ℓ)-*sparse* if $i(X) \leq k|X| - \ell$ for every subset $X \subseteq V$ with $|X| \geq 2$. It can easily be checked that the (k, ℓ)-sparse subgraphs of a given graph form the independent sets of a matroid, which is called a (k, ℓ)-*count-matroid* (see [19]). The circuits of these matroids, that are the minimal no (k, ℓ)-sparse graphs, are the graphs with exactly $k|V| - \ell + 1$ edges whose every proper subgraph is (k, ℓ)-sparse. Let K_n denote the complete graph on n vertices. If G is a circuit in the (k, ℓ)-count-matroid of $K_{|V|}$ we call it a (k, ℓ)-*circuit*. Following [1] we call a (2, 3)-circuit a *generic circuit*. The term is based on the fact that by Laman's theorem [13], (2, 3)-circuits are isomorphic to the circuits of the 2 dimensional generic rigidity matroid.

A well-known result of Nash-Williams [14] says that a graph G is decomposable into k spanning forests if and only if $i(X) \leq k(|X| - 1)$ for all nonempty subsets $X \subseteq V$. From this theorem it follows that every generic circuit is decomposable into two spanning trees, moreover we obtain the following proposition.

Proposition 1.1. *A graph G is a generic circuit if and only if it can be decomposed into two spanning trees such that no pair of proper subtrees, except single vertices, spans the same vertex set.*

Having this result one may ask whether a tree decomposition with some special properties exists. Graver et al. posed the following problem.

Open Question 1.2 (*[7], Exercise 4.69*)**.** *Does every generic circuit with vertices of degrees* 3 *and* 4 *only, have a two tree decomposition into two paths?*

Note that the smallest $(k, k+1)$-circuit is K_{2k} since a graph with $k|V| - k$ edges cannot be simple if it has less than $2k$ vertices. Recall the definition of balanced graphs from the abstract. A balanced $(k, k+1)$-circuit has vertices with degrees $2k - 1$ and $2k$ only and the number of vertices with degree $2k - 1$ is exactly $2k$.

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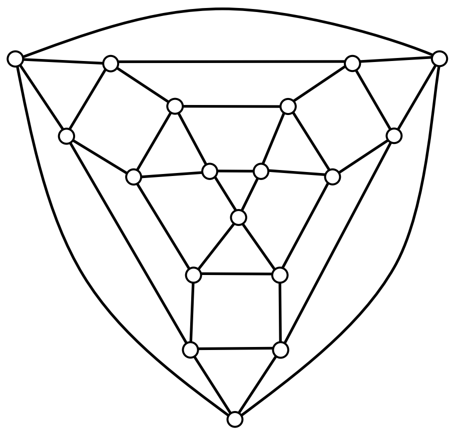

Fig. 1. The graph given by Grünbaum and Malkevitch.

In this note we show balanced $(k, k+1)$-circuits for all $k \geq 2$ which do not contain a Hamiltonian path. Thus a balanced generic circuit does not always have a decomposition demanded in Open Question 1.2. Moreover, we have a stronger result on the length of the longest paths in balanced $(k, k+1)$-circuits. For a graph G, let $h(G)$ and $h^*(G)$ denote the length of a maximum cycle and the number of vertices in a maximum path of G, respectively. Following [8] we define the *shortness exponent* $\sigma(\mathcal{G})$ for a family $\mathcal{G}$ of graphs, as follows.

$$\sigma(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \frac{\log h(G)}{\log |V(G)|}.$$

Concerning paths instead of cycles we also define the parameter $\sigma^*(\mathcal{G})$ in a similar way but with $h^*(G)$ in place of $h(G)$. Let $\mathcal{C}^{bal.}_{k,k+1}$ denote the family of balanced $(k, k+1)$-circuits. What we prove is as follows.

Theorem 1.3. $\sigma^*(\mathcal{C}^{bal.}_{k,k+1}) \leq \frac{\log 8}{\log 9}$ *for all* $k \geq 2$.

2. Preliminaries

Let $G = (V, E)$ be a graph. For $X, Y \subseteq V$, let $E(X, Y)$ denote the set of edges with one endpoint in X and another in Y. For a subset $X \subset V$, $E(X, V - X)$ is the *edge-cut* corresponding to X and $d_G(X) := |E(X, V - X)|$. We also say that X corresponds to the edge-cut. If $2 \leq |X| \leq |V| - 2$ then the edge-cut is *nontrivial*. We say that a graph G is *essentially k-edge-connected*, if all nontrivial edge-cuts of G contain at least k edges.

Our construction will be based on the observation that balanced $(k, k+1)$-circuits are just the $2k$-regular essentially $(2k+2)$-edge-connected graphs minus a vertex in the following sense.

Lemma 2.1. (i) *Let $G = (V, E)$ be a balanced $(k, k+1)$-circuit. If we add a new vertex s to G and connect it to the vertices of G with degree $2k-1$ then the obtained graph $G' = (V', E')$ is $2k$-regular and essentially $(2k+2)$-edge-connected.*
(ii) *Let $G' = (V', E')$ be a $2k$-regular essentially $(2k+2)$-edge-connected graph. If we omit an arbitrary vertex s of G' then the obtained graph $G = (V, E)$ is a balanced $(k, k+1)$-circuit.*

Proof. (i) It is clear that G' is $2k$-regular. Suppose that it is not essentially $(2k+2)$-edge-connected. Then there is a subset $X \subseteq V'$, $2 \leq |X| \leq |V'| - 2$ with $d_{G'}(X) \leq 2k$. (As G' is an Eulerian graph $d_{G'}(X) = 2k+1$ cannot hold.) We may assume that $s \notin X$. Then $2i_G(X) = 2i_{G'}(X) = (\sum_{v\in X} d_{G'}(v)) - d_{G'}(X) \geq 2k|X| - 2k$, a contradiction.
(ii) Let $X \subseteq V$ be a subset with $2 \leq |X| \leq |V| - 1$. Then the same calculation as above admits that $i_G(X) = i_{G'}(X) \leq k|X| - k - 1$. □

We call the graph G' obtained from a balanced $(k, k+1)$-circuit G as described in Lemma 2.1 (i) the *underlying regular graph* of G.

In the case of balanced generic circuits we obtain 4-regular essentially 6-edge-connected graphs. Now we can easily prove that not all balanced generic circuits can be decomposed into two Hamiltonian paths. It is clear that if a balanced generic circuit G admits such a decomposition, then the four end-vertices of the two Hamiltonian paths must be disjoint and can only be the four vertices with degree 3. Thus in the underlying 4-regular graph G' they extend to a decomposition into two Hamiltonian cycles. Therefore, it is sufficient to show a 4-regular essentially 6-edge-connected graph which does not have a decomposition into two Hamiltonian cycles.

Grünbaum and Malkevitch [9] gave an example for a 4-regular 4-connected planar graph without a Hamiltonian decomposition (see Fig. 1). One can easily check that it is essentially 6-edge-connected therefore this is a good example