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Note

A simpler proof for vertex-pancyclicity of squares of connected claw-free graphs

Alexandru I. Tomescu *

Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206, 33100 Udine, Italy
Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei, 14, 010014 Bucharest, Romania

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ABSTRACT

We give a simpler proof of the well-known result of Matthews and Sumner stating that squares of connected claw-free graphs are vertex-pancyclic. Contrary to the previous proof, our approach does not resort to Fleischner's result stating that, when restricted to squares of graphs, vertex-pancyclicity and Hamiltonicity are equivalent. The same proof idea already yielded that connected claw-free graphs of even order have a perfect matching, which is another result of Sumner. We conclude by observing that this proof identifies a larger collection of graphs for which the two properties in question hold.

1. Introduction and preliminaries

Given a graph G, we say that a vertex, or an edge, of G is *pancyclic* if it belongs to a cycle of length ℓ, for every $3 \leqslant \ell \leqslant |V(G)|$. If every vertex of G is pancyclic, then G is called *vertex-pancyclic*. A graph G is said to be *claw-free* if it does not contain an induced subgraph isomorphic to the *claw* $K_{1,3}$. The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ in which two vertices are adjacent if their distance in G is one or two. A major result about squares, due to Fleischner, states that the square of a graph is Hamiltonian if and only if it is vertex-pancyclic [2]. Matthews and Sumner showed that squares of connected claw-free graphs with at least three vertices are Hamiltonian [4], and then used Fleischner's result to conclude that they are also vertex-pancyclic. In [5] we gave simpler proofs of the facts that connected claw-free graphs (i) have a Hamiltonian cycle in their square, and (ii) admit a perfect matching if they have an even order. Notice, incidentally, that this proof method led to versions of these two proofs whose degree of formalization was so rigorous that their correctness could be verified with an automatic proof-checker, as reported in [6,7].

In this paper we elucidate the mathematical insight behind these proofs, by showing the stronger property of vertex-pancyclicity of squares of connected claw-free graphs, without resorting to Fleischner's result. We conclude by observing that this proof method identifies a larger collection of graphs for which the two properties in question hold.

Before presenting the main proof, we introduce some further terminology and a couple of basic properties. Given a graph G and a subset $S \subseteq V(G)$, we denote by $G - S$ the subgraph of G induced by $V(G) \setminus S$. Given a directed graph, or *digraph*, D and a vertex x of D, we denote by $N^+(x)$ the set of out-neighbors of x, and by $N^-(x)$ the set of its in-neighbors. Vertex x is called a *sink* if $N^+(x) = \emptyset$, and a *source* if $N^-(x) = \emptyset$. Note that every acyclic digraph owns a sink and a source. Given an acyclic digraph D and $x \in V(D)$, the *rank* of x is the length of the longest directed path from it to a sink of D. We will exploit the following three simple observations.

* Correspondence to: Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206, 33100 Udine, Italy.
E-mail addresses: alexandru.tomescu@uniud.it, alexandru.tomescu@gmail.com.

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Claim 1. *If G is a connected graph, and $v \in V(G)$, then G admits an acyclic orientation whose unique sink is v.*

To see this, reason by induction on $|V(G)|$, the claim being clear for $|V(G)| = 1$. Apply the inductive hypothesis to each connected component of $G - \{v\}$ to obtain acyclic orientations having neighbors of v as unique sinks. Obtain the desired orientation of G by orienting all edges incident to v as going towards v.

Claim 2. *Let G be a connected graph, let D be an acyclic orientation of G with a unique sink, and let s be a source of D. The graph $G - \{s\}$ is connected.*

Claim 3. *Let G be a connected claw-free graph, $|V(G)| \geqslant 2$, and let D be an acyclic orientation of G with a unique sink. Let x be a source of maximum rank r in D, and take $y \in N^+(x)$ of rank $r - 1$. The following properties hold:*

(i) *all vertices of $N^-(y)$ are sources in D,*
(ii) $|N^-(y)| \leqslant 2$,
(iii) *y is a source in the orientation of $G - N^-(y)$ induced by D.*

Points (i) and (iii) are immediate. To see (ii), observe that, from (i), there are no edges between the vertices in $N^-(y)$. If $|N^-(y)| \geqslant 3$, then $N^-(y) \cup \{y\}$ induces a claw in G.

2. Simpler proofs

For a clear treatment of the base inductive case in the proof of Theorem 1, let us make an abuse of notation and say that K_2 is Hamiltonian, that its Hamiltonian cycle uses twice its edge (so that its length is 2), and that its edge and both its vertices are pancyclic.

Theorem 1. *Let G be a connected claw-free graph with at least two vertices, and let D be an acyclic orientation of G with a unique sink. The square of G is vertex-pancyclic; moreover, if $S \subseteq V(G)$ denotes the set of sources of D, then for every $s \in S$ there exists an edge e_s of G, incident to s, such that:*

(i) *e_s is pancyclic in G^2, and*
(ii) *there exists a Hamiltonian cycle of G^2 containing all edges e_s.*

Proof. We reason by induction on $n = |V(G)|$. The result is immediate for $n \in \{2, 3\}$. Assume now that G is a connected claw-free graph on $n \geqslant 4$ vertices, and let D be an acyclic orientation of it with a unique sink. Denote by S_G the set of sources of D. Also, let x be a source of maximum rank r in D, and take $y \in N^+(x)$, of rank $r - 1$. Claims 2 and 3 imply that $|N^-(y)| \leqslant 2$ and that $H := G - N^-(y)$ has at least two vertices and is connected. Apply the inductive hypothesis to H and to its acyclic orientation with a unique sink induced by D, and consider a cycle C_H of H^2 satisfying condition (ii). Denote by S_H the set of sources of this orientation of H, and observe that $y \in S_H$, by Claim 3. Let thus $yz \in E(H)$ be the edge of C_H incident to y and pancyclic in H^2 (hence $z \notin S_H$, since $y \in S_H$).

We show that we can extend the cycle C_H to G^2, depending on the cardinality of $N^-(y)$, so that every vertex of G becomes pancyclic in G^2. Furthermore, in order to show conditions (i) and (ii) for G, observe that $N^-(y) \subseteq S_G \subseteq N^-(y) \cup (S_H \setminus \{y\})$. Accordingly, we will show that C_H can also be extended to a cycle C_G for G^2, so that, in particular, every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source s in $S_H \setminus \{y\}$, is present in C_G and is pancyclic in G^2.

If $N^-(y) = \{x\}$, we put $e_x := xy \in E(G)$. Observe first that $xz \in E(G^2)$. Let C_G be the cycle of G^2, of length $|V(G)|$, obtained from C_H by replacing the edge yz with the path yxz; see Fig. 1(b).

- Proof of (i) for G. First, e_x is pancyclic in G^2: $xyzx$ is a 3-cycle in G^2; moreover, yz is pancyclic in H^2, and thus every cycle in H^2 of length ℓ, $\ell \leqslant |V(G)| - 1$, containing yz can be extended to a cycle of length $\ell + 1$ in G^2 by replacing the edge yz with the path yxz. Second, cycle C_G includes every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$, thus every such edge is pancyclic also in G^2.
- Proof of (ii) for G. As just argued, the Hamiltonian cycle C_G of G^2 contains e_x, and every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$.
- Proof of vertex-pancyclicity of G^2. All vertices of H are pancyclic in G^2, since they are pancyclic in H^2 and they also belong to C_G. The claim follows from the fact that x belongs to the edge e_x, which is pancyclic in G^2.

If $N^-(y) = \{x, w\}$, then from the claw-freeness of G and the fact that $xw \notin E(G)$ at least one of the edges xz or wz belongs to $E(G)$, say wz. We put $e_x := xy \in E(G)$ and $e_w := wz \in E(G)$. Observe that $xw, xz \in E(G^2)$. Let C'_G and C_G be the cycles of G^2, of length $|V(G)| - 1$, and $|V(G)|$, respectively, obtained from C_H by replacing the edge yz with the path yxz, and with $yxwz$, respectively; see Fig. 1(a).

- Proof of (i) for G. First, e_x and e_w are pancyclic in G^2: $xyzx$, $wyzw$ are 3-cycles in G^2; moreover, yz is pancyclic in H^2, and thus every cycle of H^2 of length ℓ, $\ell \leqslant |V(G)| - 2$, containing yz can be extended to a cycle of length $\ell + 1$ in G^2 by replacing the edge yz with one of the paths yxz, or ywz. Cycle C_G, containing both e_x, and e_w, shows that e_x and e_w are pancyclic in G^2. Second, cycles C'_G and C_G include every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$, thus every such edge is pancyclic also in G^2.