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The Möbius function of generalized factor order

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ABSTRACT

We use discrete Morse theory to determine the Möbius function of generalized factor order. Ordinary factor order on the Kleene closure A^* of a set A is the partial order defined by letting $u \le w$ if w contains u as a subsequence of consecutive letters. Generalized factor order takes into account a partial order P_A on the alphabet A, that is, $u \le w$ whenever w contains a subsequence $w(i+1)\cdots w(i+|u|)$ such that for each j, $u(j) \le w(i+j)$ in A. Using Babson and Hersh's application of Robin Forman's discrete Morse theory to poset order complexes, we are able to give a recursive formula for the Möbius function in the case where each element of A covers a unique letter in P_A.

1. Introduction

The Möbius function of ordinary factor order was determined by Björner [2]. Having reproved his formula using discrete Morse theory (see [7]), we wished to investigate whether this result would generalize to a wider class of posets. In this paper, we determine the formula for the Möbius function when factor order is generalized to include a partial ordering P of letters, provided each letter covers a unique element in P. We also show the formula we obtain is a generalization of Björner's formula. Since it is clear our formula would have been nearly impossible to discover using other techniques of investigating Möbius functions, this paper illustrates the ability of discrete Morse theory to simplify complex combinatorial problems of this nature.

We begin with a brief introduction to ordinary factor order. Let A be any set. The *Kleene closure*, A^*, is the set of all finite length words over A. So if w is a word and $w(i)$ is the ith letter in w, then

$$A^* = \{w = w(1)\cdots w(n) : 0 \le n < \infty \text{ and } w(i) \in A \text{ for all } i\}.$$

The *length* of w, denoted $|w|$, is the number of letters in w. *Ordinary factor order* on A^* is the partial order on A^* defined by letting $u \le w$ if w contains a subsequence of consecutive letters $w(i+1)\cdots w(i+n)$ such that $u(j) = w(i+j)$ for $1 \le j \le n = |u|$. When $u \le w$, we call u a *factor* of w. A word u is *flat* if $u(1) = \cdots = u(n)$, where $n = |u|$.

A *prefix* of a word $w \in A^*$ is a factor of w that includes the first letter of w. Similarly, a *suffix* of w is a factor of w that contains the last letter of w. A prefix or suffix is *proper* if it is not equal to w. Define the *outer word* $o(w)$ of w to be the longest factor that appears as both a proper prefix and suffix in w. Notice that $o(w)$ can be the empty word. Define the *inner word* $i(w)$ of w to be the factor $i(w) = w(2)\cdots w(n-1)$, where $n = |w|$.

The following theorem of Björner gives a formula for the Möbius function in ordinary factor order.

Theorem 1.1 ([2]). *In ordinary factor order, if $u \le w$ then*

$$\mu(u,w) = \begin{cases} \mu(u, o(w)) & \text{if } |w|-|u| > 2 \text{ and } u \le o(w) \not\le i(w), \\ 1 & \text{if } |w|-|u| = 2,\ w \text{ is not flat, and } u = o(w) \text{ or } u = i(w), \\ (-1)^{|w|-|u|} & \text{if } |w|-|u| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

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Babson and Hersh developed a useful tool for investigating Möbius functions through the maximal chains of an interval [1]. Their result gives a way of applying Forman's discrete version of Morse theory [3] to partially ordered sets. For brevity, we are only going to state the minimum number of definitions to apply this theorem. A reader who is interested in learning more about discrete Morse theory is encouraged to begin with Robin Forman's outstanding introduction to the topic [4].

Let P be any poset. The notation $y \to x$ will indicate that y covers x in P. Let $C = (z_0 \to \cdots \to z_n)$ be a chain in P. Notice that z_0 is the maximum element in this chain and z_n is the minimum element, so that each step in the chain is a step down in the poset. Since each pair of adjacent elements are related by a cover, C is called a *saturated chain*. It will often be useful for us to think of the chain C as a subset of the poset P. For example, since a *maximal chain* is maximal under set containment, C is maximal chain in P if and only if z_0 is a maximal element of P and z_n is a minimal element of P.

Define the *closed interval* of a chain C from z_i to z_j to be the chain $C[z_i, z_j] = (z_i \to \cdots \to z_j)$. The *open interval* of C from z_i to z_j, $C(z_i, z_j)$, is defined similarly. For simplicity, the closed interval $C[z_i, z_i]$ consisting of the single element z_i will also be written z_i, but the context will always indicate whether we are referring to the element or the interval. Also, since our chains are listed top to bottom, an interval $C[y, x]$ is non-empty when $y \geq x$ in the poset P, while an interval $[x, y]$ in the poset is non-empty when $x \leq y$ in P. Notice that C is a maximal chain in $[x, y]$ if and only if $z_0 = y$ and $z_n = x$.

Given two maximal chains $C = (z_0 \to \cdots \to z_n)$ and $D = (x_0 \to \cdots \to x_n)$ in an interval $[x, y]$, we say C and D *agree to index k* if $z_i = x_i$ for all $i \leq k$. We say C and D *diverge from index k* if C and D agree to index k and $z_{k+1} \neq x_{k+1}$. A total ordering $C_1 < \cdots < C_n$ of the maximal chains of an interval is a *poset lexicographic order* if it satisfies the following: suppose $C < D$ and C and D diverge from index k; if C' and D' agree to index $k+1$ with C and D, respectively, then $C' < D'$.

Suppose $C_1 < \cdots < C_n$ is an ordering of the maximal chains of the closed interval $[x, y]$. An interval $C(z_i, z_j)$ is a *skipped interval* of a maximal chain C if

$$C - C(z_i, z_j) \subseteq C' \quad \text{for some } C' < C.$$

It is a *minimal skipped interval (MSI)* if it does not properly contain another skipped interval. We write $I(C)$ for the set of all MSIs of a chain C. To find the set $I(C)$, first consider each interval $I \subseteq C(y, x)$ and determine if $C - I \subseteq C'$ for any $C' < C$, then throw out any such interval that is nonminimal. In Tables 3.4 and 3.8, we give examples of minimal skipped intervals in the context of generalized factor order.

Notice $I(C)$ could contain intervals which overlap, that is, intervals with non-empty intersection. Babson and Hersh's result requires a set of disjoint intervals derived from $I(C)$, which we will denote $J(C)$. We construct $J(C) = \{J_1, J_2, \ldots\}$ as follows. Order the intervals of $I(C)$ based on when they are first encountered in C. Recall that since our chains are viewed top-down, larger elements in P have smaller indices in C. Thus, I_1 will contain the element z_i of smallest index that appears in any interval in $I(C)$, I_2 will contain the element $z_j < z_i$ of smallest index that appears in any interval in $I(C)$ other than I_1, etc. Let $J_1 = I_1$. Consider the intervals $I_2' = I_2 - J_1$, $I_3' = I_3 - J_1$, and so forth. Throw out any that are no longer minimal in the set $I_2', I_3', \ldots$, and pick the first one that remains to be J_2. Continue this process until no intervals remain to add to $J(C)$.

The set of intervals $J(C)$ *covers* C if its union equals the open interval $C(y, x)$. A chain C is called *critical* if $J(C)$ covers C. Finally, when a chain C is critical, the *critical dimension* of the chain is

$$d(C) = \#J(C) - 1$$

where # denotes cardinality.

Theorem 1.2 (*[1]*)**.** *For any poset lexicographic order on the maximal chains of* $[x, y]$,

$$\mu(x, y) = \sum_C (-1)^{d(C)},$$

where the sum is over all critical chains C in the poset lexicographic order. □

The rest of the paper is organized as follows. The next section will give a brief summary of our results. In Section 3, we will consider in detail generalized factor order on the integers. Section 4 considers generalized factor order on rooted forests, and gives a formula for the Möbius function which contains the formula of Section 3 and Björner's formula as subcases. This is not obvious, and an independent proof is needed to establish the connection with Björner's formula. Section 5 discusses open problems related to this work.

2. Summary of results

To use Babson and Hersh's Theorem 1.2, we first define a lexicographic order on the maximal chains of any interval. Next, we characterize the corresponding MSIs. The final step is creating a useful description of the critical chains in $J(C)$. In this particular case, our description of the critical chains requires us to introduce a number of definitions before we can find the contribution of each critical chain to the Möbius function using Theorem 1.2.

In this section, we record the major definitions in this paper and use them to state our formula for the Möbius function in its most general form.

Let F be a rooted forest. A *flat* word in the Kleene closure F^* is a sequence of r's, where r is a minimal element in F. If a word w in F^* is not flat, then a letter $w(i)$ is reducible if $i = 1$, $i = |w|$, or $w(i)$ is not a minimal element in F.