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Mutually describing multisets and integer partitions

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ABSTRACT

To each finite multiset A, with underlying set $S(A)$, we associate a new multiset $d(A)$, obtained by adjoining to $S(A)$ the multiplicities of its elements in A. We study the orbits of the map d under iteration, and show that if A consists of nonnegative integers, then its orbit under d converges to a cycle. Moreover, we prove that all cycles of d over $\mathbb{Z}$ are of length at most 3, and we completely determine them. This amounts to finding all systems of mutually describing multisets. In the process, we are led to introduce and study a related discrete dynamical system on the set of integer partitions of n for each $n \geq 1$.

1. Introduction

John H. Conway's "look-and-say sequence" [1]

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, . . .

is notable for mixing the concepts of *numeral* and *cardinal number*. The sequence starts with 1, and each subsequent element is obtained from the preceding one by describing, from left to right, what number one sees and with what frequency it appears. For instance, the description of 1211 is "one 1, one 2 and two 1's". Converting the frequencies "one" and "two" into the numerals 1 and 2, respectively, we get the next sequence element, namely 111221. See entry A005150 in [9,2,6,11] for related works.

In the present paper, we consider an analogous type of sequence, starting with a given multiset and iteratively producing new multisets by similarly converting, at each stage, element frequencies (*cardinals*) into elements proper (*numerals*).

For instance, let us start with the multiset $A_0 = [1, 1, 2, 2, 5, 5, 5]$. Then A_0 may be described by the two-row array

$$\delta(A_0) = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 3 \end{pmatrix},$$

where the first row gives the underlying set $\{1, 2, 5\}$ of A_0, and the second row the respective multiplicities in A_0 of its distinct elements. We now view $\delta(A_0)$ *as a multiset in itself*, by considering all of its entries as elements proper. This yields the new multiset $A_1 = [1, 2, 5, 2, 2, 3] = [1, 2, 2, 2, 3, 5]$, and we shall accordingly write

$$A_1 = d(A_0).$$

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Now A_1 is itself described by the two-row array

$$\delta(A_1) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 1 & 1 \end{pmatrix},$$

yielding in turn the new multiset $A_2 = [1, 2, 3, 5, 1, 3, 1, 1] = [1, 1, 1, 1, 2, 3, 3, 5]$, and so on. It is quite natural to wonder how this sequence evolves in the long run. Well, as the reader may check, the 6th iterate $A_6 = d^6(A_0)$ is equal to

$$A_6 = [1, 1, 1, 2, 2, 3, 3, 3, 4, 5],$$

with descriptor array

$$\delta(A_6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 1 & 1 \end{pmatrix}.$$

Now this array, when viewed as a multiset, is A_6 itself again! In other words, A_6 is a fixed point of d and, equivalently, a *self-describing multiset*.

There also are longer cycles under d. For instance, a 2-cycle is realized by the multisets $A_0 \rightleftarrows A_1$ where

$$\delta(A_0) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}, \qquad \delta(A_1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 1 & 1 \end{pmatrix}.$$

In such a situation, each of the two multisets describes the other one, or more precisely, underlies the descriptor array of the other one.

Our aim in the present paper is to study the dynamical behavior of the map d, i.e., to describe the long-term behavior of its orbits.[1] Of particular interest is the quest for its fixed points and cycles, since they correspond to systems of mutually describing multisets. The main results of this paper read as follows.

1. *If A is a finite multiset over $\mathbb{N} = \{0, 1, 2, \ldots\}$, then the orbit of A under d converges to a cycle*. (See Theorem 3.1.)
2. *All cycles of d over $\mathbb{Z}$ are of length at most* 3. (See Theorem 5.4.)

In contrast, the first statement does not hold over $\mathbb{Z}$. For instance, the orbit of $A = [-1, 0]$ under d is divergent. (See top of Section 3.) Going beyond the second statement, we shall classify all cycles of d over $\mathbb{Z}$ in Theorem 6.2.

A few earlier papers have been concerned with related notions of self- or mutually describing strings, such as the string

$$a_0a_1 \ldots a_9 = 6210001000,$$

where a_i gives the multiplicity of i for all $i = 0, 1, \ldots, 9$. See for instance [4,5,7,12]. See also the final note preceding the references.

The paper is organized as follows. We start with a formal definition of d and basic remarks in Section 2. In Section 3, we prove orbit convergence for finite multisets over $\mathbb{N}$. In Section 4, we introduce a related dynamical system on the set of integer partitions of n for $n \geq 1$, and describe all of its limit cycles. We then prove in Section 5 that the cycles of d over $\mathbb{Z}$ have length at most 3. These cycles, finally, are fully classified in Section 6.

2. Generalities

In this section, we gather a few notations, define the map d more formally, and make basic remarks used throughout the paper. Let $A = [a_1, \ldots, a_N]$ be a finite multiset over $\mathbb{Z}$, of *cardinality* $|A| = N$. We shall denote by $S(A)$ the underlying set of A, by $\mu_A(x)$ the multiplicity in A of any $x \in A$, and by $M(A)$ the *multiplicities multiset* of A, i.e.,

$$M(A) = [\mu_A(x)]_{x \in S(A)}.$$

For example, if $A_0 = [1, 1, 2, 2, 5, 5, 5]$, then $S(A_0) = \{1, 2, 5\}$ and $M(A_0) = [2, 2, 3]$. We define the *length* of A as the cardinality of its underlying set,

$$l(A) = |S(A)|.$$

If A is of length n, with $S(A) = \{b_1, \ldots, b_n\}$ and $M(A) = [\mu_1, \ldots, \mu_n]$, where $\mu_i = \mu_A(b_i)$ for all i, we associate to A the $2 \times n$ integer array

$$\delta(A) = \begin{pmatrix} b_1 & \cdots & b_n \\ \mu_1 & \cdots & \mu_n \end{pmatrix},$$

which we call the *descriptor array* of A: its first row gives the distinct elements of A, and its second row their respective multiplicities in A. Note that $\delta(A)$ is only defined up to a permutation of its columns. We are now in a position to define our map d on multisets: we set

$$d(A) = [b_1, \ldots, b_n, \mu_1, \ldots, \mu_n].$$

[1] The map d and the problem of describing its dynamics first appeared, mistakenly as an exercise, in [3, pp. 38–39].