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Overlarge sets of Mendelsohn triple systems with resolvability[☆]

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ABSTRACT

An OLRMTS(v) (OLARMTS(v)) over a $(v+1)$-set X is a collection $\{(X \setminus \{x\}, \mathcal{B}_x) : x \in X\}$ of $v+1$ pairwise disjoint resolvable (almost resolvable) Mendelsohn triple systems of order v. In this paper several direct construction methods for OLRMTSs and OLARMTSs are presented and then applied to produce some new orders; the smallest unknown OLRMTS(v) for $v = 18, 24$, the smallest unknown OLARMTS(v) for $v = 19, 22, 28, 31$, and some other small designs are displayed; a few new existence families are also obtained by known recursive constructions.

1. Introduction

Let X be a finite set. In what follows an *ordered pair* of X is always an ordered pair (x, y) where $x \neq y \in X$. A *cyclic triple* on X is a set of three ordered pairs (x, y), (y, z) and (z, x) of X, which is denoted by $\langle x, y, z\rangle$ (or $\langle y, z, x\rangle$, $\langle z, x, y\rangle$).

A *Mendelsohn triple system* of order v (MTS(v)) is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of cyclic triples on X, called *blocks*, such that every ordered pair of X belongs to exactly one block of $\mathcal{B}$.

For a v-set X, a set P of some cyclic triples on X is said to be a *parallel class* if P forms a partition of X. P is said to be an *almost parallel class* if P forms a partition of $X \setminus \{x\}$ for some $x \in X$.

An MTS(v) is called *resolvable* (or *almost resolvable*) if its block set $\mathcal{B}$ can be partitioned into parallel classes (or almost parallel classes); the collection of all the parallel classes (or almost parallel classes) is called a *resolution*. A resolvable MTS(v), denoted by RMTS(v), is easily checked to contain $v-1$ parallel classes. An almost resolvable MTS(v), denoted by ARMTS(v), contains v almost parallel classes, each missing a distinct element.

Let X be a set of $v+1$ elements. An *overlarge set* of MTS(v)s, denoted by OLMTS(v), is a collection $\{(X \setminus \{x\}, \mathcal{B}_x) : x \in X\}$, where every $(X \setminus \{x\}, \mathcal{B}_x)$ is an MTS(v) and $\bigcup_{x\in X} \mathcal{B}_x$ forms a partition of all cyclic triples on X. An OLRMTS(v) (or OLARMTS(v)) denotes an OLMTS(v) in which each MTS(v) is resolvable (or almost resolvable).

Lemma 1.1 (*[1,2,6]*).

(1) *An* RMTS(v) *exists if and only if* $v \equiv 0 \pmod 3$ *and* $v \neq 6$. *An* ARMTS(v) *exists if and only if* $v \equiv 1 \pmod 3$.
(2) *An* OLMTS(v) *exists if and only if* $v \equiv 0, 1 \pmod 3$ *and* $v \neq 6$.

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The research on overlarge sets of resolvable or almost resolvable Mendelsohn triple systems is very limited. Kang and Tian [7] posed their existence problem and presented some preliminary results. Some research on overlarge sets of Kirkman triple systems has been done, from which we can derive OLRMTSs immediately; see for instance [8–11].

Lemma 1.2 (*[3,7,11]*).

(1) *There exists an* OLRMTS(v), *where*
  (a) $v = 12, 21$,
  (b) $v = 3(2^{2p+1}25^q + 1)$ *for* $p + q \geq 1$, *or*
  (c) $v = \prod_{i=1}^{s_1}(2q_i^{n_i} + 1)\prod_{j=1}^{s_2}(4^{a_j}25^{b_j} - 1)$, *where* $s_1 + s_2 \geq 1$, $s_1, s_2, b_j \geq 0$, $n_i, a_j \geq 1$, $q_i \equiv 7 \pmod{12}$ *and* q_i *is a prime power.*

(2) *There exists an* OLARMTS(v) *for* $v = 10, 4^n, 7^n, 13^n, 25^n, 4^n25$, *where* $n \geq 1$.

In this paper we present some direct construction methods for OLRMTSs and OLARMTSs and then, with the aid of computer search, we produce a number of new orders; we get the smallest unknown OLRMTS(v) for $v = 18, 24$, the smallest unknown OLARMTS(v) for $v = 19, 22, 28, 31$, and some other small designs; a few new existence families are also obtained by known recursive constructions.

2. Direct construction methods

Let S_v denote the symmetric group on a v-set X. Let $X^{\langle 3\rangle}$ denote the set of all $2\binom{v}{3}$ cyclic triples on X. For a cyclic triple $T = \langle x, y, z\rangle$ on X and a permutation $\alpha \in S_v$, define $\alpha(T) = \langle \alpha(x), \alpha(y), \alpha(z)\rangle$. Suppose G is a permutation group of X (a subgroup of S_v). For $A, B \in X^{\langle 3\rangle}$, define $A \sim B$ if $\alpha(A) = B$ for some $\alpha \in G$. Then $\sim$ is an equivalence relation on $X^{\langle 3\rangle}$. Each equivalence class is called an *orbit* of $X^{\langle 3\rangle}$ with respect to the group G. If the cardinality of an orbit is equal to the order of G, the orbit is called *full*, or *short* otherwise. Obviously all orbits comprise a partition of $X^{\langle 3\rangle}$.

Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two (A)RMTS(v)s possessing resolutions Γ_1 and Γ_2 respectively. They are *isomorphic* if there exists a bijection $\alpha : X_1 \to X_2$ such that $\alpha(\mathcal{B}_1) = \mathcal{B}_2$ and $\alpha(\Gamma_1) = \Gamma_2$, where $\alpha(\mathcal{B}) = \{\alpha(B) : B \in \mathcal{B}\}$ and $\alpha(\Gamma) = \{\alpha(P) : P \in \Gamma\}$. An *automorphism* is an isomorphism of an (A)RMTS(v) with itself.

In this paper we do not discuss OLRMTS(v) for $v \equiv 3 \pmod 6$ because for such a congruence class the research on overlarge sets of Kirkman triple systems (OLKTS) would be more challenging and an OLKTS(v) implies an OLRMTS(v), which was mentioned in Section 1 (see [8–11] for constructions of OLKTSs). We will investigate direct constructions of OLRMTS$(q-1)$s for $q \equiv 1 \pmod 6$ or OLARMTS $(q-1)$s for $q \equiv 5 \pmod 6$ over the point set F_q, where q is a prime power and F_q is the finite field of q elements. It is possible to construct an OL(A)RMTS$(q-1)$ over the finite field F_q consisting of q (A)RMTS$(q-1)$s which are isomorphic to one another under the action of $\alpha_i : x \mapsto x + i$ for some $i \in F_q$. We call any one of the isomorphic (A)RMTS$(q-1)$ a *base system*. We will also construct directly a number of OLARMTS(q)s over $F_q \cup \{\infty\}$ for prime powers $q \equiv 1 \pmod 6$. For such a case we hope that an OLARMTS(q) consists of q isomorphic ARMTS(q)s under the action of the additive group F_q and an extra non-isomorphic ARMTS(q). We now illustrate an example to show the existence of the smallest unknown OLRMTS(v) for $v = 18$.

Lemma 2.1. *There exists an* OLRMTS(18).

Proof. Take Z_{19} as the point set. Let $Q_0 = \{r\langle 1, 7, 11\rangle : r \in R\}$ and $Q_0' = \{r\langle 1, 11, 7\rangle : r \in R\}$ where $R = \{1, 2, 4, 5, 8, 10\}$. Then Q_0 and Q_0' form two parallel classes of $Z_{19}^* = Z_{19} \setminus \{0\}$. We need fifteen further parallel classes to form an RMTS(18) over Z_{19}^*, listed as follows, each line forming a parallel class.

⟨3, 4, 5⟩	⟨2, 9, 16⟩	⟨14, 6, 17⟩	⟨10, 11, 8⟩	⟨13, 1, 18⟩	⟨15, 7, 12⟩
⟨1, 2, 4⟩	⟨15, 3, 7⟩	⟨5, 6, 14⟩	⟨11, 18, 9⟩	⟨10, 12, 16⟩	⟨13, 8, 17⟩
⟨7, 14, 9⟩	⟨10, 2, 11⟩	⟨16, 4, 3⟩	⟨1, 12, 6⟩	⟨15, 18, 5⟩	⟨13, 17, 8⟩
⟨11, 3, 6⟩	⟨13, 14, 1⟩	⟨17, 9, 2⟩	⟨7, 8, 4⟩	⟨15, 5, 18⟩	⟨10, 16, 12⟩
⟨6, 7, 10⟩	⟨4, 11, 13⟩	⟨9, 1, 15⟩	⟨17, 18, 3⟩	⟨5, 12, 2⟩	⟨16, 8, 14⟩
⟨14, 15, 1⟩	⟨7, 18, 17⟩	⟨8, 9, 3⟩	⟨2, 16, 13⟩	⟨10, 5, 4⟩	⟨11, 6, 12⟩
⟨3, 10, 7⟩	⟨11, 12, 5⟩	⟨18, 6, 2⟩	⟨14, 17, 15⟩	⟨13, 16, 9⟩	⟨1, 4, 8⟩
⟨2, 13, 11⟩	⟨1, 8, 16⟩	⟨12, 4, 14⟩	⟨3, 5, 10⟩	⟨15, 17, 6⟩	⟨7, 9, 18⟩
⟨12, 13, 3⟩	⟨8, 15, 2⟩	⟨18, 10, 14⟩	⟨5, 1, 6⟩	⟨16, 7, 4⟩	⟨17, 11, 9⟩
⟨9, 10, 1⟩	⟨5, 8, 7⟩	⟨12, 14, 13⟩	⟨18, 16, 6⟩	⟨17, 3, 11⟩	⟨4, 2, 15⟩
⟨6, 13, 7⟩	⟨16, 18, 11⟩	⟨8, 3, 15⟩	⟨12, 17, 4⟩	⟨5, 2, 1⟩	⟨9, 14, 10⟩
⟨4, 15, 11⟩	⟨17, 12, 1⟩	⟨18, 2, 10⟩	⟨8, 5, 9⟩	⟨16, 14, 7⟩	⟨6, 3, 13⟩
⟨15, 16, 11⟩	⟨10, 17, 1⟩	⟨13, 5, 7⟩	⟨6, 8, 2⟩	⟨4, 18, 14⟩	⟨9, 12, 3⟩
⟨18, 1, 3⟩	⟨12, 7, 2⟩	⟨8, 11, 14⟩	⟨15, 6, 16⟩	⟨10, 4, 17⟩	⟨13, 9, 5⟩
⟨17, 2, 7⟩	⟨5, 14, 11⟩	⟨16, 3, 1⟩	⟨13, 18, 4⟩	⟨15, 12, 9⟩	⟨10, 8, 6⟩

It is readily checked that developing the given RMTS(18) under the additive operation of Z_{19} yields an OLRMTS(18). □