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Reduced criteria for degree sequences

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ABSTRACT

For many types of graphs, criteria have been discovered that give necessary and sufficient conditions for an integer sequence to be the degree sequence of such a graph. These criteria tend to take the form of a set of inequalities, and in the case of the Erdős–Gallai criterion (for simple undirected graphs) and the Gale–Ryser criterion (for bipartite graphs), it has been shown that the number of inequalities that must be checked can be reduced significantly. We show that similar reductions hold for the corresponding criteria for many other types of graphs, including bipartite r-multigraphs, bipartite graphs with structural edges, directed graphs, r-multigraphs, and tournaments. We also prove a reduction for imbalance sequences.

1. Introduction

There is a family of results that give necessary and sufficient conditions for an integer sequence to be the degree sequence of a given type of graph. A well-known example is the Erdős–Gallai criterion [12]: given $d \in \mathbb{Z}^n$ such that $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ and $\sum d_i$ is even, there exists a simple undirected graph on n vertices with degrees $d_1, \ldots, d_n$ if and only if

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

for all $k \in \{1, \ldots, n\}$. It is natural to ask: Are these conditions minimal? For example, must one check the inequality for all $k \in \{1, \ldots, n\}$? It turns out that the answer is "no". It was shown by Zverovich and Zverovich [24], and later by Tripathi and Vijay [23], that it is sufficient that the inequality hold for $k = m$ and for all $k < m$ such that $d_k > d_{k+1}$, where $m = \max\{i : d_i \geq i\}$. As a result, it is possible to reduce the number of inequalities to the cardinality of $\{d_1, \ldots, d_n\}$ or less. This type of reduction enjoys both theoretical and practical utility. On the theoretical side, it facilitates further results relating to degree sequences. On the practical side, savings in computation can be realized for algorithms dealing with large graphs.

In the same spirit as the Erdős–Gallai result, a spectrum of degree sequence criteria have been discovered for diverse classes of graphs, including bipartite graphs [15,22], bipartite r-multigraphs [5], bipartite graphs with structural edges [2], directed graphs [13,7], r-multigraphs [7,9], tournaments [18], and imbalance sequences of directed graphs [21]. Given the benefits of the reduction described above, it would be desirable to obtain analogous reductions for these other criteria as well.

The purpose of this paper is to show that such reductions can indeed be obtained for all these types of graphs. Most of these results appear to be new. For those results that are old, we provide new proofs, since our approach provides clear

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intuitions about why they are true. Using basic notions from finite calculus and some observations about convex sequences, we find it possible to solve these problems in a unified way, leading to proofs that are exceedingly simple and highly interpretable.

In previous work, other authors have studied the problem of degree sequence criterion reduction in three of the classes of graphs addressed in this paper.

- In the case of simple undirected graphs, several researchers [11,19,24,23,10,3] have noticed that the Erdős–Gallai criterion can be reduced. The strongest general result among these is due to Zverovich and Zverovich [24], who obtained a further improvement to the reduction described above. Zverovich and Zverovich [24] also proved the interesting fact that all of the Erdős–Gallai inequalities are satisfied if the length of the sequence exceeds a certain bound that depends only on the maximum and minimum degrees. Barrus, Hartke, Jao, and West [3] recently extended this result to sequences with bounded gaps between degrees. Dahl and Flatberg [10] made the insightful observation that concave sequences play an essential role in reducing the Erdős–Gallai criterion. It turns out that this observation can be vastly extended to cover many other classes of graphs as well (when appropriately generalized to "almost concave" sequences).
- In the case of bipartite graphs, the celebrated Gale–Ryser theorem [15,22] provides a degree sequence criterion of a similar form. Zverovich and Zverovich [24] also address this case, proving a reduction similar to the one described above for simple undirected graphs. In the present work, we obtain a stronger reduction, via a much simpler proof.
- In the case of tournaments, Landau [18] provided a degree sequence criterion that can also be reduced in a similar manner, as noted (without proof) by Beineke [4]. We prove an even stronger reduction.

The paper is organized as follows. Section 2 contains definitions and elementary results. In Section 2.1, we state some standard definitions and facts from finite calculus. In Section 2.2, we introduce the essential notion of an almost concave sequence, and make a few elementary observations regarding concave and almost concave sequences. Section 2.3 demonstrates the utility of almost concavity by giving a short new proof of a well-known theorem of Fulkerson and Ryser. The main results of the paper are in Section 3, where after some general remarks we prove a series of reduced degree sequence criteria, covering many classes of graphs. Section 3 ends with a negative result: a "counterexample" illustrating that these reductions are nontrivial in the sense that they do not hold for every class of graphs with a degree sequence criterion of the type described above.

2. Concave and almost concave sequences

It turns out that the reductions to be proved in Section 3 hinge upon certain properties that can be succinctly and intuitively described using finite calculus. All of the reductions can be proven without using finite calculus, but our experience is that it yields dividends, both conceptually and notationally.

2.1. A brief refresher on finite calculus

In this subsection we state some standard definitions and results from finite calculus [16]. Given $a \in \mathbb{R}^n$, define $\nabla a \in \mathbb{R}^n$ by

$$(\nabla a)_k = a_k - a_{k-1}$$

for $k \in \{1, \ldots, n\}$, where by convention $a_0 = 0$. (The sequence ∇a is sometimes referred to as the "backward difference", as opposed to the "forward difference": $(\Delta a)_k = a_{k+1} - a_k$. For our purposes, it is more notationally convenient to work with backward differences, but of course all the results below could be restated in terms of forward differences.) We often find it preferable to use the following alternative notation: define

$$\dot{a} = \nabla a, \quad \text{and}$$
$$\ddot{a} = \nabla(\nabla a).$$

The following elementary identities may be easily verified.

Proposition 2.1 (*Basic Properties*)**.** *Suppose $a, b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and $m \in \mathbb{Z}$ with $m \geq 1$. Then*

1. $\nabla(a + b) = \nabla a + \nabla b$
2. $\nabla(ca) = c(\nabla a)$
3. $a_k - a_j = \sum_{i=j+1}^{k} \dot{a}_i$ *whenever* $1 \leq j < k \leq n$.

2.2. Concave and almost concave sequences

In this subsection, we introduce the essential notion of an "almost concave" sequence, and make a few elementary observations regarding concave and almost concave sequences. A sequence $a \in \mathbb{R}^n$ is said to be *concave* if

$$\ddot{a}_k \leq 0 \quad \text{for all } k \text{ s.t. } 3 \leq k \leq n.$$