Discrete Mathematics 312 (2012) 1077–1083

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Upper bounds on minimum balanced bipartitions[☆]

Genghua Fan [a], Baogang Xu [b,*], Xingxing Yu [c], Chuixiang Zhou [a]

[a] Center for Discrete Mathematics, Fuzhou University, Fuzhou, 350002, China
[b] School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210046, China
[c] School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

ARTICLE INFO

Article history:
Received 23 January 2010
Received in revised form 25 October 2011
Accepted 28 November 2011
Available online 22 December 2011

Keywords:
Balanced bipartition
Upper bound
Plane graph

ABSTRACT

A balanced bipartition of a graph G is a partition of $V(G)$ into two subsets V_1 and V_2, which differ in size by at most 1. The minimum balanced bipartition problem asks for a balanced bipartition V_1, V_2 of a graph minimizing $e(V_1, V_2)$, where $e(V_1, V_2)$ is the number of edges joining V_1 and V_2. We present a tight upper bound on the minimum of $e(V_1, V_2)$, giving one answer to a question of Bollobás and Scott. We prove that every connected triangle-free plane graph G of order at least 3 has a balanced bipartition V_1, V_2 with $e(V_1, V_2) \leq |V(G)| - 2$, and we show that $K_{1,3}$, $K_{3,3} - e$, and $K_{2,n}$, with $n \geq 1$, are precisely the extremal graphs. We also show that every plane graph G without separating triangles has a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \leq |V(G)| + 1$.

1. Introduction

Graph partition problems ask for a partition of the vertex set of a graph into pairwise disjoint subsets with various requirements. Given a partition $V_1, \ldots, V_k$ of $V(G)$, we use $e(V_i)$ to denote the number of edges with both ends in V_i, and let $e(V_1, \ldots, V_k) = |E(G)| - \sum_{i=1}^{k} e(V_i)$ ($e(V_1, \ldots, V_k)$ is usually called the *size* of the partition). Readers are referred to [6] for notation and terminology.

A classical example of partition problems is the *maximum bipartite subgraph problem*: given a graph G, find a partition V_1, V_2 of $V(G)$ that maximizes $e(V_1, V_2)$.

Let V_1, V_2 be a bipartition of $V(G)$. Edwards [7,8] proved that every graph with m edges admits a bipartition V_1, V_2 such that $e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}\sqrt{2m + \frac{1}{4}} - \frac{1}{8}$. This bound is best possible, holding with equality for the complete graphs K_{2n+1}. Bollobás and Scott [3] generalized Edward's result to k-partitions and showed that, for each integer $k \geq 1$, every graph G with m edges admits a vertex partition $V_1, \ldots, V_k$ such that the number of edges with ends in distinct subsets is at least

$$\frac{k-1}{k}m + \frac{k-1}{2k}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right) - \frac{k^2 - 4k + 4}{8k}.$$

The bound is again best possible, as shown by the complete graph K_{kn+1}.

In contrast to the problem of finding a partition $V_1, \ldots, V_k$ maximizing $e(V_1, \ldots, V_k)$, Bollobás and Scott [1,2] considered the problem of finding a partition $V_1, \ldots, V_k$ minimizing $\max\{e(V_i) : i = 1, \ldots, k\}$. This is a "judicious" partition problem, as it asks for a partition to optimize several quantities simultaneously. Bollobás and Scott [2] showed that every graph

☆ Partially supported by NSFC 10931003 and 11171160.
* Corresponding author.
E-mail address: baogxu@njnu.edu.cn (B. Xu).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.11.030

with m edges admits a bipartition V_1, V_2 such that $e(V_1, V_2) \geq m/2 + (\sqrt{2m+1/4} - 1/2)/4$ and $\max\{e(V_1), e(V_2)\} \leq m/4 + (\sqrt{2m+1/4} - 1/2)/8$. Xu and Yu [15,16] generalized this result to k-partitions by showing that, for any integer $k \geq 1$ and for any graph G with m edges, $V(G)$ admits a partition $V_1, \ldots, V_k$ such that $e(V_1, \ldots, V_k) \geq (k-1)m/k + (k-1)(\sqrt{2m+1/4} - 1/2)/(2k) + O(k)$ and $\max\{e(V_i)\} \leq m/k^2 + (k-1)(\sqrt{2m+1/4} - 1/2)/(2k^2)$.

A partition $V_1, \ldots, V_k$ is said to be *balanced* if the sizes of the sets differ by at most 1. Balanced bipartition problems of weighted graphs are usually referred to as *bisection problems*. The *maximum bisection problem* (respectively, *minimum bisection problem*) asks for a balanced bipartition V_1, V_2 maximizing (respectively, minimizing) the sum of the weight on the edges joining V_1 and V_2. It is easy to see that, for unweighted graphs, the maximum bisection problem and the minimum bisection problem are equivalent (by considering complements). Both problems are NP-complete [9], and they have been studied extensively from the algorithmic perspective because of their extensive applications. The maximum bisection problem for plane graphs was shown to be NP-hard by Jerrum, while the complexity of the minimum bisection problem for plane graphs remains unknown (see [10]).

As Bollobás [4] pointed out, the extremal problems for balanced partitions have been relatively little investigated; there are even no bounds analogous to that of Edwards for the maximum bipartite subgraph problem. Bollobás and Scott [5] proved that almost every regular graph with m edges admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leq m/4$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and minimum degree of graph G, respectively. Xu et al. [13] extended the method used by Bollobás and Scott in [5] and proved that, for any graph G with m edges and $\Delta(G) \leq 7\delta(G)/5$, and for every balanced bipartition V_1, V_2 of $V(G)$ maximizing $e(V_1, V_2)$, we have $\max\{e(V_1), e(V_2)\} \leq m/3$. In [14], Xu et al. prove, by employing a different counting technique, that every graph with m edges and minimum degree at least 5 admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leq m/3$, while a conjecture of Bollobás and Scott [4] claims that every graph with minimum degree at least 2 admits such a bipartition.

In [4], Bollobás and Scott asked the following.

Problem 1.1. For a graph G with n vertices and m edges, what are the largest and smallest cuts that we can guarantee with balanced bipartitions?

In [14], Xu et al. showed that a graph G with m edges admits a balanced bipartition of size at least $\frac{m+|M|}{2}$, where M is a maximum matching of G. (The existence of such a bipartition without requiring balance is well known. See p. 37 of [11].) This bound is sharp on the complete graph K_{2n+1}. We use G^c to denote the complement of a graph G. With a similar argument to that of [14], we prove an upper bound on minimum balanced bipartitions of graphs.

Theorem 1.2. *Let M be a maximum matching in G^c of a graph G that has n vertices and m edges. Then G admits a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \leq \frac{1}{2}(m + \lfloor \frac{n}{2} \rfloor - |M|)$.*

The bound of Theorem 1.2 is also sharp, as the equality holds on complete graphs. Together with the above-mentioned lower bound on maximum balanced bipartitions in [14], it gives one answer to Problem 1.1. It is still an open question to find a function $f(m)$ (respectively, $g(m)$) such that every graph on m edges admits a balanced bipartition with at least $\frac{m}{2} + f(m)$ (respectively, at most $\frac{m}{2} + g(m)$) edges joining the two subsets.

A folklore conjecture claims that every plane graph of order n has a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \leq n$. This conjecture, if true, is best possible, as shown by K_4 (in fact, we will present an infinite family of such plane graphs).

In Section 3, we consider connected triangle-free plane graphs, prove an upper bound on minimum balanced bipartition of such graphs, and characterize the extremal graphs. Let $K_{3,3} - e$ denote the graph obtained from $K_{3,3}$ by removing an edge.

Theorem 1.3. *Every connected triangle-free plane graph of order $n \geq 3$ has a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \leq n - 2$. The extremal graphs are precisely $K_{1,3}$, $K_{3,3} - e$, and $K_{2,k}$, $k \geq 1$.*

A triangle T in a connected plane graph G is called a *separating triangle* if both the interior and the exterior of G are not empty. In Section 4, we prove the following Theorem 1.4 on minimum balanced bipartition of plane graphs without separating triangles.

Theorem 1.4. *Let G be a plane graph of order n. If G contains no separating triangles, then G admits a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \leq n + 1$.*

Let G be a graph, x be a vertex of G, and S be a subset of $V(G)$. We use $N_S(x)$ to denote the set of neighbors of x in S.

2. An upper bound for all graphs

In this section, we will prove Theorem 1.2, which gives a tight upper bound on the minimum size of a balanced bipartition.

Proof of Theorem 1.2. Let $M = \{u_1v_1, \ldots, u_rv_r\}$, and let $U = \{u_1, \ldots, u_r, v_1, \ldots, v_r\}$. Note that $G - U$ is a complete graph if $U \neq V(G)$. First, we choose $V_1^{(0)}, V_2^{(0)}$ to be an arbitrary balanced bipartition of $V(G) \setminus U$ such that $|V_1^{(0)}| \geq |V_2^{(0)}|$. Then, let $V_1^{(i)}$ and $V_2^{(i)}$, for i from 1 to r, be obtained from $V_1^{(i-1)}$ and $V_2^{(i-1)}$ such that

(a) $V_1^{(i-1)} \subseteq V_1^{(i)}$ and $V_2^{(i-1)} \subseteq V_2^{(i)}$,