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Factorizations of complete graphs into brooms

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ABSTRACT

Let r and n be positive integers with $r < 2n$. A broom of order $2n$ is the union of the path on P_{2n-r-1} and the star $K_{1,r}$, plus one edge joining the center of the star to an endpoint of the path. It was shown by Kubesa (2005) [10] that the broom factorizes the complete graph K_{2n} for odd n and $r < \lfloor \frac{n}{2} \rfloor$. In this note we give a complete classification of brooms that factorize K_{2n} by giving a constructive proof for all $r \leq \frac{n+1}{2}$ (with one exceptional case) and by showing that the brooms for $r > \frac{n+1}{2}$ do not factorize the complete graph K_{2n}.

1. Introduction and definitions

Graph decomposition is a well established topic of graph theory. Various techniques were introduced for decomposing graphs into edge disjoint subgraphs.

Definition 1.1. Let H be a graph with m vertices. A *decomposition* of the graph H is a set of pairwise edge disjoint subgraphs $G_1, G_2, \ldots, G_s$ of H such that every edge of H belongs to exactly one of the subgraphs G_r. If each subgraph G_r is isomorphic to a graph G we speak about a *G-decomposition of H*. If G is a factor (i.e., a spanning subgraph) of H, then we call the G-decomposition a *G-factorization*.

In this paper we always take the complete graph K_m for H and a certain spanning tree T for G. There are some obvious necessary conditions for a T-factorization of K_m to exist. First, since the number of edges $m-1$ of T must divide the number of edges $m(m-1)/2$ of K_m, obviously m has to be even and there will be $m/2$ copies of T in the factorization. Moreover, since every vertex has degree at least 1 in every factor, $\Delta(T) \leq m/2$. Further structure-based necessary conditions are examined in Section 2.

Unlike the famous Graceful Tree Conjecture (all n-vertex trees have graceful labellings, which enable them to decompose K_{2n}), not all $2n$-vertex trees factorize K_{2n}. There is no easy necessary and sufficient condition known for a T-factorization to exist, and we do not expect such a condition to exist.

Sufficient conditions include several types of graph labellings. If a given graph G allows a certain type of labeling, then there exists a G-factorization of K_{2n}. One such labeling, the blended labeling, was introduced by Frŏnček [3]. A fundamental notion in further constructions is the length of an edge.

We adopt the common convention of denoting vertices by their labels. Moreover, we denote an edge xy by (x, y) if x or y are integer expressions.

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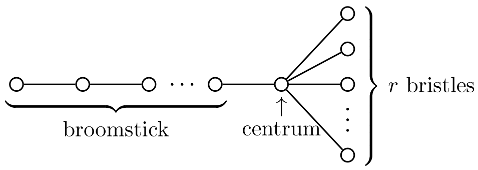

Fig. 1. A broom $B_{2n}(r)$.

Definition 1.2. Let G be a graph with $V(G) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = m$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (m-1)_i\}$ for both $i = 0$ and $i = 1$.

The *pure length* of an edge (x_i, y_i) with $x_i, y_i \in V_i$, where $i \in \{0, 1\}$, for $\lambda(x_i) = p_i$ and $\lambda(y_i) = q_i$ is defined as

$$\ell_{ii}(x_i, y_i) = \min\{|p-q|,\ m - |p-q|\}.$$

The *mixed length* of an edge (x_0, y_1) with $x_0 \in V_0, y_1 \in V_1$, for $\lambda(x_0) = p_0$ and $\lambda(y_1) = q_1$, is defined as

$$\ell_{01}(x_0, y_1) = \begin{cases} q-p & \text{for } q \geq p \\ m+q-p & \text{for } q < p, \end{cases}$$

where p and q are the vertex labels without subscripts and lie in $\{0, 1, \ldots, m-1\}$. The edges (x_i, y_i) for $i \in \{0, 1\}$ with the pure length ℓ_{ii} are *pure edges* and the edges (x_0, y_1) with the mixed length ℓ_{01} are *mixed edges*.

Definition 1.3. Let G be a graph with $4n+1$ edges such that $V(G) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n+1$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (2n)_i\}$ for both $i = 0$ and $i = 1$, and define lengths as in Definition 1.2.

We say G has a *blended labeling* (also called a *blended ρ-labeling*) λ if

(1) $\{\ell_{ii}(x_i, y_i) \colon (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$ for $i = 0, 1$,
(2) $\{\ell_{01}(x_0, y_1) \colon (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n\}$.

Fronček [3] showed that there exists a G-factorization of K_{2n} for odd n if G has a blended labeling. Meszka [11] showed that having a blended labeling is not necessary for a G-factorization to exist when G is a tree. Kovářová [8] (see also [4]) introduced 'swapping labeling' and showed that a G-factorization of K_{2n} for even n exists when G has a swapping labeling.

Definition 1.4. Let G be a graph with $V(G) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (2n-1)_i\}$ for both $i = 0$ and $i = 1$, and define lengths as in Definition 1.2.

We say that G with $4n-1$ edges has a *swapping blended labeling* (briefly *swapping labeling*) λ if

(1) $\{\ell_{ii}(x_i, y_i) \colon (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$, for $i = 0, 1$,
(2) there exists an isomorphism φ such that G is isomorphic to G', where $V(G') = V(G)$ and $E(G') = E(G) \setminus \{(k_0, (k+n)_0), (l_1, (l+n)_1)\} \cup \{(k_0, (l+n)_1), ((k+n)_0, l_1)\}$ for certain k, l,
(3) $\{\ell_{01}(x_0, y_1) \colon (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n-1\} \setminus \{\ell_{01}(k_0, (l+n)_1)\}$.

We summarize the results by Fronček [3] and Kovářová [8] in the following theorem.

Theorem 1.5. *If a graph G on m vertices allows a blended labeling or a swapping labeling, then there exists a G-factorization of K_m.*

Various other labellings such as the ρ-symmetric labeling, $2n$-cyclic labeling, fixing labeling, and recursive labeling were introduced by several authors as sufficient conditions for certain G-factorizations to exist. For every admissible $d \geq 3$, a spanning tree of diameter d that factorizes K_{4n+2} was found by Fronček in [3]; the case for K_{4n} was completed by Kovářová in [8]. Among the most general result there is the determination of spanning caterpillars of diameter 4 that factorize K_{2n} (in a series of papers by Fronček [2], Kubesa [10,9] and Kovářová [6,7]). Spanning caterpillars of diameter 5 that factorize K_{2n} were determined through the years in a series of papers and finally completed in [4] by Fronček et al.. A T-factorization of K_{2n} for every $\Delta(T)$ possible, $2 \leq \Delta(T) \leq n$, was given by Kovář and Kubesa [5].

In this paper we give a complete characterization (analogously to the papers [4,5]) in the case when a tree consisting of a path with many leaves attached to one of its endvertices factorizes the corresponding complete graph. We show that every such graph factorizes the corresponding complete graph unless the number of attached leaves exceeds $(n+1)/2$ or unless it is one exceptional case. The primary motivation for studying this class was to examine graphs that do not have labellings of the types mentioned and yet do factorize the complete graph. It turned out that there were only finitely many such graphs in this particular class of graphs.

Let S_r denote the star $K_{1,r}$, and let P_k denote a path with k vertices. For $1 \leq r \leq 2n-3$, let $B_{2n}(r)$ denote the graph formed from the disjoint union of P_{2n-r-1} and S_r by adding one edge joining the center of the star to an endvertex of the path. The graph $B_{2n}(r)$ is called a *broom*, the center of the star is called the *centrum*, the leaves of the star are called *bristles*, the path is the *broomstick*, and its vertices are called *broomstick vertices*. See Fig. 1.

We seek a decomposition of K_{2n} into n factors $T_1, T_2, \ldots, T_n$ that are isomorphic to a single spanning tree T, where $T \cong B_{2n}(r)$. Using the labeling of the vertices of K_{2n}, we designate the factors by isomorphisms $\phi_1, \phi_2, \ldots \phi_n$, writing $T_i = \phi_i(B_{2n}(r))$. This is an abuse of notation; actually, ϕ_i is the map of the vertex set.