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Chop vectors and the lattice of integer partitions

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ABSTRACT

The set $\mathscr{S}$ consists of all finite sets of integer length sticks. By listing the lengths of these sticks in nonincreasing order, we can represent each element S of $\mathscr{S}$ by a nonincreasing sequence of positive integers. These sequences can then be partially ordered by dominance to obtain a lattice (also denoted by $\mathscr{S}$) closely related to the lattice of integer partitions. The *chop vector* of an element $S \in \mathscr{S}$ is defined to be the infinite vector $\mathbf{v}_S = (v_1, v_2, v_3, \ldots)$, where each v_w is the minimum number of cuts needed to chop S into unit pieces, given a knife which can cut up to w sticks at a time. The chop vectors are ordered componentwise. In this paper, we show that the mapping that takes any element of $\mathscr{S}$ to its chop vector is order-preserving.

1. Introduction

The purpose of this paper is to prove a theorem (Theorem 1.1) that connects two previously unrelated topics: a result in [4] on parallel cutting of integer-length sticks, and the lattice of integer partitions. Theorem 1.1 answers a question discussed in [3].

In [4], Ginsburg and Sands considered the following problem. Suppose we are given a finite set of sticks of positive integer lengths. We wish to chop these sticks into unit-length pieces, using a knife that can cut up to w sticks at a time, where w is a fixed positive integer (called the *width* of the knife). How should we proceed in order to chop up the sticks using as few cuts as possible?

The solution in [4] goes as follows: at each step, choose the w longest *nontrivial* (that is, of length greater than one) sticks, or all nontrivial sticks if there are less than w of them, and chop these all in half or as nearly in half as possible (that is, each stick of even length $2n$ is cut into two sticks of length n, while each stick of odd length $2n+1$ is cut into sticks of lengths n and $n+1$). This natural algorithm (called the *binary* algorithm in [4]) not surprisingly turns out to minimize the number of chops needed in all cases. (Incidentally, in her thesis, [3] the first author has proven that the above algorithm can be weakened slightly while still being an optimal solution for this problem.[2])

We will identify a set of k sticks with an infinite non-increasing sequence S of positive integers, where the first k integers in S represent the lengths of the sticks, and the remaining members of S are all 1's. The set of all such sequences S will be denoted by $\mathscr{S}$. Note that the addition (or deletion) of 1's (which represent trivial sticks not needing to be cut) at the end of any $S \in \mathscr{S}$ will not affect the number of chops needed. Thus, for example, $(5, 2, 2, 1, 1, \ldots)$ will usually be denoted by $(5, 2, 2)$.

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[1] Some of the material in this paper is contained in the thesis [3] of the first author.

[2] Namely, for *odd* n, sticks of length $2n$ can be cut into sticks of lengths $n-1$ and $n+1$ instead of into equal length-n sticks. However, in this paper we will use the original binary algorithm.

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We can define a partial order $\leq$ (called *dominance*) on $\mathscr{S}$ by: for all $S = (s_1, s_2, \ldots)$ and $T = (t_1, t_2, \ldots)$ in $\mathscr{S}$,

$$S \leq T \quad \text{if and only if } \sum_{i=1}^{j} s_i \leq \sum_{i=1}^{j} t_i \text{ for all } j \geq 1.$$

Then $(\mathscr{S}, \leq)$ becomes a lattice, which we also denote simply by $\mathscr{S}$. There is a close relationship between the lattice $\mathscr{S}$ and the lattice of integer partitions, which we give in the next section.

For each $S \in \mathscr{S}$, define the infinite vector $\mathbf{v}_S = (v_1, v_2, v_3, \ldots)$ where, for $w \geq 1$, v_w is the minimum number of cuts needed to chop S into unit pieces given a knife which can cut up to w pieces at a time. We call $\mathbf{v}_S$ the *chop vector* of S.

Note that v_1 is the number of cuts required to chop all nontrivial sticks in S into units, one stick at a time, and so $v_1 = \sum_{s \in S}(s-1)$. Also, the chop vector is a non-increasing sequence of non-negative integers, so $\mathbf{v}_S$ is eventually constant.

For example, consider $S = (7, 3, 2)$, for which $v_1 = 9$. Also $v_2 = 5$, since the binary algorithm with a knife of width $w = 2$ cuts S in five steps as follows:

$$(7,3,2) \to (4,3,2,2,1) \to (2,2,2,2,2,1,1) \to (2,2,2,1,1,1,1,1)$$
$$\to (2,1,1,1,1,1,1,1,1,1,1) \to (1,1,1,1,1,1,1,1,1,1,1,1).$$

Ignoring trivial sticks, we write this dissection as

$$(7,3,2) \to (4,3,2,2) \to (2,2,2,2,2) \to (2,2,2) \to (2) \to \emptyset.$$

But with a knife of width $w = 3$, the dissection takes only three steps:

$$(7,3,2) \to (4,3,2) \to (2,2,2) \to \emptyset.$$

Moreover, it is easy to see that, for width at least 3, at least three cuts are needed. Thus $\mathbf{v}_{(7,3,2)} = (9, 5, 3, 3, \ldots)$.

The family of all chop vectors, considered as elements of the direct product $\mathbb{N}^{\omega}$, can be naturally ordered componentwise; that is, for all $S, T \in \mathscr{S}$, put $\mathbf{v}_S \leq \mathbf{v}_T$ if and only if $(\mathbf{v}_S)_i \leq (\mathbf{v}_T)_i$ for all i.

We can now state our main theorem.

Theorem 1.1. *The function $\phi : \mathscr{S} \to \mathbb{N}^{\omega}$ defined by $\phi(S) = \mathbf{v}_S$ for all $S \in \mathscr{S}$ is order-preserving; that is, if $S \leq T \in \mathscr{S}$, then $\mathbf{v}_S \leq \mathbf{v}_T$.*

The next section contains some background information. The proof of Theorem 1.1, along with a lemma that forms a large part of the proof, is given in Section 3. Some closing remarks appear in the last section.

2. Preliminaries

If we deduct 1 from each entry in a sequence $S \in \mathscr{S}$, we obtain an infinite non-decreasing sequence S' of non-negative integers, only finitely many of which are nonzero. Thus we will consider S' as a partition of the positive integer $\sum_{s' \in S'} s' = \sum_{s \in S}(s-1)$. Therefore $\{S' : S \in \mathscr{S}\}$ forms the set $\mathscr{P}$ of all integer partitions (see for example [2,1,7]). Furthermore, $\mathscr{P}$ can be given a natural partial ordering $\leq$ called *dominance ordering* (or *majorization*) as follows. For integer partitions $S = (s_1, s_2, \ldots)$ and $T = (t_1, t_2, \ldots)$ in $\mathscr{P}$, put $S \leq T$ if and only if $\sum_{i=1}^{j} s_i \leq \sum_{i=1}^{j} t_i$ for all $j \geq 1$.

Under the dominance ordering, $\mathscr{P}$ is a lattice (called the *lattice of integer partitions*), as shown in [1] and in [7], though in [1] the lattice is denoted by *NPL*, and in [7], the definition of $\leq$ is slightly different so that the lattice obtained (denoted by $L_B(\infty)$ in [7]) is dually isomorphic to $\mathscr{P}$. Much earlier, Brylawski [2] had considered the partitions of a fixed integer n, under dominance ordering, but did not combine the resulting finite lattices of integer partitions into the full (infinite) lattice $\mathscr{P}$. Other papers dealing briefly with the finite lattices of integer partitions are [6,8] and the survey paper [5].

In our introduction, the ordering we defined on the set $\mathscr{S}$ was also dominance ordering. Clearly, the lattices $\mathscr{S}$ and $\mathscr{P}$ are isomorphic, via the renaming $S \to S'$. Fig. 1 (adapted from a diagram for $\mathscr{P}$ in [1]) shows the lower part of the lattice $\mathscr{S}$, containing all elements $S \in \mathscr{S}$ satisfying $v_1 \leq 8$. As has already been mentioned, we suppress the 1's from the elements of $\mathscr{S}$, so that only nontrivial sticks are shown.

The following result follows immediately from the analogous result for $\mathscr{P}$ (see [2,1] and also Chapter 5, Lemma D.1 in [9]).

Lemma 2.1. *Dominance ordering $\leq$ on the lattice $\mathscr{S}$ is the transitive and reflexive closure of the following two types of relations. Consider $S, T \in \mathscr{S}$ with $S = (s_1, \ldots, s_m)$ and $T = (t_1, \ldots, t_n)$. It follows that $S < T$ if*

(i) *$n = m + 1$, $t_{m+1} = 2$, and $s_i = t_i$ for all $1 \leq i \leq m$, or*
(ii) *$n = m$ and there exist j and k with $1 \leq j < k \leq m$ such that $t_j = s_j + 1$, $t_k = s_k - 1$, and $t_i = s_i$ for all $i \neq j$ or k.*

Considering S and T as (multi)sets of (lengths of) sticks rather than as nonincreasing sequences of lengths,

- (i) *is equivalent to $T = S \cup \{2\}$, and*
- (ii) *is equivalent to $T = (S - \{x, y\}) \cup \{x - 1, y + 1\}$ for some x, y in S satisfying $2 \leq x \leq y$.*

For instance, $(4, 3) < (4, 3, 2)$ in Fig. 1 is an example of the first kind of relation above, while $(4, 3, 2) < (5, 2, 2)$ and $(4, 3, 2) < (4, 4, 1) = (4, 4)$ are examples of the second kind. Thus $(4, 3) < (5, 2, 2)$ and $(4, 3) < (4, 4)$ in the transitive closure.