Discrete Mathematics 312 (2012) 1201–1212

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Nash equilibria and values through modular partitions in infinite games

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ARTICLE INFO

Article history:
Received 6 April 2010
Received in revised form 6 December 2011
Accepted 7 December 2011
Available online 2 January 2012

Keywords:
Game theory
Non cooperative games
Zero-sum symmetric two-player game
Nash equilibrium
Value
Partition
Module
Oriented graph
Tournament

ABSTRACT

Quotients through modular partitions are well adapted to the computation of Nash equilibria and values in the (zero-sum, two-player) game associated to a bounded skew-symmetric matrix. Applications to infinite oriented graphs are provided.

1. Introduction

In November 1713, in a letter to Nicolas Bernoulli, the Marquis Pierre Rémond de Montmort (see "*Essay d'analyse sur les jeux de hazard*", [7, p. 406]) proposed the following game which seemed unsolvable to him: "A father wants to give a new year's gift to his son and says to him that he will hide in his hand a token with a natural number, "even" or "odd". If his son says "even" (respectively, "odd") and the token is "even" (respectively, "odd"), then he will give 4 (respectively, 1) ecus to his son; else he will give 0 ecu to his son". Then Montmort asks several questions. Which rules should be prescribed to the father (respectively, to the son) in order that he loses (respectively, earns) a little (respectively, a lot of) money? How much money will the son earn if each player follows the most advantageous way for him? Various mathematicians studied extensions of this game, now called (finite) zero-sum two-player games in "normal form". They proposed "solutions" now called *equilibria* using *mixed strategies*. In the Montmort example, the most advantageous way for each player is to play "even" with probability $\frac{1}{5}$, and "odd" with probability $\frac{4}{5}$; then the expected payoff of the son is $\frac{4}{5}$. In the particular case of finite *symmetric* zero-sum two-player games with at most five *pure strategies*, Borel [2–4] showed the existence of (at least) an equilibrium. Among these games, the "Rock–Paper–Scissors" game or its extension "Rock–Paper–Scissors–Well" (see Example 1) are widely known. Later, von Neumann [19,18] and Nash (see [16,17]) proved the existence of "Nash equilibria" in every *finite n*-player game. Notice that generally such equilibria are not unique; however, some uniqueness theorems have been proved for certain symmetric two-player zero-sum finite games (see [11–13]).

The study of *infinite* two-player zero-sum games is more difficult since the "minimax theorem" for an arbitrary bounded matrix does not hold anymore (consider, for example, the matrix of the reverse (usual) strict order on $\mathbb{N}$ without any greatest

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.12.005

element [20, p. 182]), so Nash's theorem cannot be applied to infinite zero-sum two-player games in normal form. However, some results for infinite games were obtained: for example, Méndez-Naya ([14, Th. 4.3 p. 228], [15, Th. 4 p. 84]) showed the existence of a *value* for certain bounded infinite matrices in which every row converges.

In this paper, we consider the notion of *modular partition* for a bounded skew-symmetric (finite or infinite) matrix M. Denoting by Q the quotient matrix of M through a modular partition, we then prove that every Nash equilibrium of (the zero-sum, two-player game associated to) M allows to compute a Nash equilibrium of Q (Theorem 1), and also, Nash equilibria of restrictions of M to the modules of positive weight with respect to the equilibrium of M (Theorem 2). A converse statement providing Nash equilibria of M knowing Nash equilibria of Q and of the restrictions of M also holds (Theorem 3). Notice that this converse statement is linked to the *composition consistency* property in finite tournaments (see [11]). We then prove that, if M has a value, Q also has a value (Theorem 4). We define the *upper value* of a bounded matrix, and, we show (Theorem 5) that the upper value of M is the upper value of $Q + D$, where D is the diagonal matrix of the upper values of the (restrictions of M to its) modules. Finally, we describe various examples of computations of Nash equilibria or upper values in infinite oriented graphs.

The paper is organized as follows. In Section 2, we recall *Nash equilibria* of the game associated to a (finite or infinite) bounded real matrix; in Section 3 we define *modules* and *modular partitions* of a skew-symmetric matrix; in Sections 4 (respectively, 5 and 6) we prove theorems for Nash equilibria (respectively, values and upper values) of a skew-symmetric bounded matrix M in the context of modular partitions; finally, in Section 7, we apply our results to various oriented graphs.

2. Nash equilibria of bounded matrices

2.1. Bounded matrices

Given a set I, $\ell^1(I)$ denotes the space of $X = (x_i)_{i\in I} \in \mathbb{R}^I$ such that $\|X\|_1 := \sum_{i\in I} |x_i| < +\infty$, and $\ell^\infty(I)$ denotes the space of $X = (x_i)_{i\in I} \in \mathbb{R}^I$ such that $\|X\|_\infty := \sup_{i\in I} |x_i| < +\infty$. Given sets I, J, a real matrix $M = (a_{i,j})_{i\in I, j\in J}$ is *bounded* if $\|M\|_\infty := \sup_{i\in I; j\in J} |a_{i,j}| < +\infty$. Every bounded matrix M defines a continuous linear mapping from $\ell^1(J)$ to $\ell^\infty(I)$, associating to every $Y = (y_i)_{i\in J} \in \ell^1(J)$ (viewed as a one-column matrix) the element $MY := (\sum_{j\in J} a_{i,j}y_j)_{i\in I}$ of $\ell^\infty(I)$; notice that $\|MY\|_\infty \le \|M\|_\infty \|Y\|_1$. We denote by $\langle ., .\rangle : \ell^1(I) \times \ell^\infty(I) \to \mathbb{R}$ the continuous bilinear mapping associating to every $X = (x_i)_{i\in I} \in \ell^1(I)$ and every $Y = (y_i)_{i\in I} \in \ell^\infty(I)$, the real number $\sum_{i\in I} x_i y_i$. We denote by $\beta_M : \ell^1(I) \times \ell^1(J) \to \mathbb{R}$ the continuous bilinear mapping associating to each $X \in \ell^1(I)$ and $Y \in \ell^1(J)$ the real number $\langle X, MY\rangle$ (also denoted by $X^T MY$, where X^T is the transposed matrix of the one-column matrix X). If the bounded matrix $M = (a_{i,j})_{i,j\in I}$ is *skew symmetric* ($M^T = -M$, i.e., for every $i, j \in I$, $a_{i,j} = -a_{j,i}$), then the bilinear mapping β_M is also skew-symmetric (for every $X, Y \in \ell^1(I)$, $\beta_M(Y, X) = -\beta_M(X, Y)$).

2.2. The (two-player) zero-sum game associated to a bounded matrix

2.2.1. Mixed strategies on a set I

Given a set I, we denote by Δ_I the set of $X = (x_i)_{i\in I} \in [0, 1]^I$ such that $\sum_{i\in I} x_i = 1$. The elements of Δ_I correspond to the *discrete probabilities* on the set I: they are also called *mixed strategies* on I. For every $i \in I$, we denote by δ_i the element of Δ_I such that $\delta_i(j) = 1$ if $j = i$, and $\delta_i(j) = 0$ if $j \neq i$. Thus δ_i corresponds to the *Dirac measure* at point i. Dirac measures at points of I are also called *pure strategies* on I.

Remark 1. In the normed space $\ell^1(I)$, the subset Δ_I is the closed convex hull of the set $\{\delta_i : i \in I\}$ of Dirac measures on I. Besides, the closed unit ball of the normed space $\ell^1(I)$ is the closed convex hull of the set $\{\pm\delta_i : i \in I\}$.

2.2.2. Payoff function associated to a bounded matrix

Given a bounded real matrix $M = (a_{i,j})_{i\in I, j\in J}$, the restriction of β_M to the set $\Delta_I \times \Delta_J$ is the *payoff* function associated to the matrix M; we denote it by G_M. Notice that the mapping $G_M : \Delta_I \times \Delta_J \to \mathbb{R}$ associates to every $(X, Y) \in \Delta_I \times \Delta_J$ the *expectation* of M according to the (discrete) probability $X \otimes Y$ on the set $I \times J$.

2.3. Saddle points

Given two sets A, B, consider a mapping $G : A \times B \to \mathbb{R}$. A point $(a, b) \in A \times B$ is a *saddle point* of G if, for every $x \in A$ and $y \in B$, $G(x, b) \le G(a, b) \le G(a, y)$.

We recall the following remarks, which are well known.

Remark 2. The set of saddle points of a real mapping $G : A \times B \to \mathbb{R}$ is a *product set*: if (a, b) and (c, d) are saddle points of G, then (c, b) and (a, d) are also saddle points of G.

Remark 3. If $G : A \times A \to \mathbb{R}$ is skew symmetric, and if (a, b) is a saddle point of G, then (b, a) is also a saddle point of G; thus (a, a) and (b, b) are saddle points of G, and, for every saddle point (a, b) of G, $G(a, b) = 0$ (because