Discrete Mathematics 312 (2012) 2146–2152

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Acyclic homomorphisms to stars of graph Cartesian products and chordal bipartite graphs

Mieczysław Borowiecki, Ewa Drgas-Burchardt *

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Z. Szafrana 4a, Zielona Góra, Poland

A R T I C L E   I N F O

Article history:
Available online 28 September 2011

Keywords:
Graph homomorphism
Hypercube
Chordal bipartite graph
Cartesian product
Feedback vertex set

A B S T R A C T

Homomorphisms to a given graph H (H-colourings) are considered in the literature among other graph colouring concepts. We restrict our attention to a special class of H-colourings, namely H is assumed to be a star. Our additional requirement is that the set of vertices of a graph G mapped into the central vertex of the star and any other colour class induce in G an acyclic subgraph. We investigate the existence of such a homomorphism to a star of given order. The complexity of this problem is studied. Moreover, the smallest order of a star for which a homomorphism of a given graph G with desired features exists is considered. Some exact values and many bounds of this number for chordal bipartite graphs, cylinders, grids, in particular hypercubes, are given. As an application of these results, we obtain some bounds on the cardinality of the minimum feedback vertex set for specified graph classes.

1. Preliminaries

We consider only finite undirected graphs without loops or multiple edges. For definitions and notations not presented here, we refer to [1,4].

Let the given graphs be G and H. A *homomorphism* of G to H is a mapping $h : V(G) \to V(H)$ such that $h(v)h(v') \in E(H)$ whenever $vv' \in E(G)$. If there exists a homomorphism of G to H, then we say that G is *homomorphic* to H and write $G \to H$. Clearly $G \to K_k$ coincides with the existence of a k-colouring of G. Thus each homomorphism h of G to H is called an H-colouring, where labels of vertices from $V(H)$ are referred to as colours. To emphasize that h is the partition of $V(G)$ into colour classes V_i (possible empty) indexed by the vertices of $V(H)$, we sometimes write $h = (V_0, \ldots, V_{|V(H)|-1})$.

The existence of an H-colouring, called the H-COLOURING PROBLEM, was thoroughly discussed in several papers. We refer to the book [12] of Hell and Nešetřil for thorough introduction to this topic. We present below the celebrated result of these authors.

Theorem 1 (*Hell, Nešetřil [11]*)**.** *The H-*COLOURING PROBLEM *is polynomial time solvable when H is bipartite and* NP*-complete otherwise.*

Let $\mathcal{Q} \subseteq \mathcal{B}$, where $\mathcal{B}$ is the class of all bipartite graphs and let there exist an H-colouring of a graph G such that for any two colour classes V_i, V_j the graph $G[V_i \cup V_j]$ induced by them belongs to $\mathcal{Q}$. Such a homomorphism is called an $(H, \mathcal{Q})$*-colouring* and its existence will be denoted by $G \overset{\mathcal{Q}}{\to} H$.

In order to preserve monotonicity of parameters associated with an $(H, \mathcal{Q})$-colouring, we assume that $\mathcal{Q}$ is a hereditary and additive property, i.e. $\mathcal{Q}$ is closed under isomorphism, taking (induced) subgraphs and if graphs, say X and Y, are in $\mathcal{Q}$, then their disjoint union is in $\mathcal{Q}$.

* Corresponding author.
E-mail addresses: m.borowiecki@wmie.uz.zgora.pl (M. Borowiecki), e.drgas-burchardt@wmie.uz.zgora.pl (E. Drgas-Burchardt).

0012-365X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.08.030

Now we are ready to formulate a new colouring problem.

$(H, \mathcal{Q})$-COLOURING PROBLEM
INSTANCE: A graph G.
QUESTION: Does there exist an $(H, \mathcal{Q})$-colouring of G?

Of course for a given graph H, an H-colouring is a particular case of an $(H, \mathcal{Q})$-colouring with $\mathcal{Q}$ being the class of all bipartite graphs. Examples of other well-known $(H, \mathcal{Q})$-colourings for $H = K_k$ are: $\mathcal{Q} = \mathcal{SF}$ (star forest), $\mathcal{Q} = \mathcal{LF}$ (linear forest), $\mathcal{Q} = \mathcal{D}_1$ (1-degenerate), i.e. star, linear and acyclic colourings, respectively. The most notable of them, a $(K_k, \mathcal{D}_1)$-colouring, was introduced by Grünbaum [9]. Referring to it, in 1978 Kostochka proved the following theorem.

Theorem 2 ([14]). *Let $k \geq 3$ be a positive integer. The $(K_k, \mathcal{D}_1)$-COLOURING PROBLEM is NP-complete.*

On the other hand, for each graph G, there exists a positive integer k such that $G \overset{\mathcal{D}_1}{\to} K_k$. The minimum k satisfying this condition is called the *acyclic chromatic number* of G and denoted by $\chi_a(G)$.

2. Star-acyclic colourings

2.1. Star-acyclic chromatic number

For k being a positive integer, let S_k denote a star of order k. In this subsection, we are focusing on $(S_k, \mathcal{D}_1)$-colourings of graphs. Obviously, each $(S_k, \mathcal{D}_1)$-colouring h of a given graph G can be viewed as a partition of $V(G)$ into colour classes V_i (possible empty) satisfying some additional requirements. Precisely, let $h = (V_0, \ldots, V_{k-1})$, be the $(S_k, \mathcal{D}_1)$-colouring of G such that $V_0, \cup_{i=1}^{k-1} V_i$ are independent sets in G and the graph $G[V_j \cup V_0]$ is acyclic for each j, $1 \leq j \leq k-1$. Recall that, if $h = (V_0, \ldots, V_{k-1})$ is the $(S_k, \mathcal{D}_1)$-colouring of G and some $v \in V_j$, then it means that $h(v) = j$.

Note that if H is a bipartite graph and $G \overset{\mathcal{Q}}{\to} H$, then G has to be bipartite. But now we need the following theorem.

Proposition 1. *For every bipartite graph $G = (X, Y; E)$, there is a positive integer k such that $G \overset{\mathcal{D}_1}{\to} S_k$ holds.*

Proof. Let $Y = \{y_1, \ldots, y_{k-1}\}$. The colouring $h = (X, \{y_1\}, \ldots, \{y_{k-1}\})$ is the $(S_k, \mathcal{D}_1)$-colouring of G. □

The smallest integer k for which a bipartite graph G has an $(S_k, \mathcal{D}_1)$-colouring is called the *star-acyclic chromatic number* of G and is denoted by $\chi_{(S,\mathcal{D}_1)}(G)$. An acyclic homomorphism of G to a star of some order is called an $(S, \mathcal{D}_1)$-colouring of G.

2.2. Complexity

To motivate our later investigation, the complexity of the following problem is studied.

$(S_k, \mathcal{D}_1)$-COLOURING PROBLEM
INSTANCE: A bipartite graph $G = (X, Y; E)$.
QUESTION: Is $\chi_{(S,\mathcal{D}_1)}(G) \leq k$?

Let G be a graph and e be its edge with end vertices u, v. The *subdivision* of e yields a graph obtained from $G - e$ by adding a new vertex w and edges uw, wv. Let $S(G)$ denote a graph obtained from G by subdividing each of its edges exactly once.

Let us denote by $va(G)$, $(ea(G))$ the *vertex arboricity* (*edge arboricity*) of G, i.e. the minimum number of parts in the partition of $V(G)$ $(E(G))$, each of which induces an acyclic subgraph in G.

Theorem 3. *For each graph G, $\chi_{(\mathcal{S},\mathcal{D}_1)}(S(G)) = va(G) + 1$.*

Proof. Let X be the set of vertices of $S(G)$ obtained by subdividing all the edges of G, i.e. $V(S(G)) \setminus X = V(G)$. First, assume that G is connected. Consider an $(S_p, \mathcal{D}_1)$-colouring h of $S(G)$. Since $S(G)$ is bipartite, then either $h = (X, V_1, \ldots, V_{p-1})$ or $h = (V(G), X_1, \ldots, X_{p-1})$. In the first case, $p-1$ has to be no smaller than $va(G)$, while in the second one $p-1$ has to be no smaller than $ea(G)$. Since $va(G) \leq ea(G)$ for each graph G [2], in both cases we have $\chi_{(\mathcal{S},\mathcal{D}_1)}(S(G)) \geq va(G)+1$. Furthermore, there exists an $(S_{va(G)+1}, \mathcal{D}_1)$-colouring h of $S(G)$ in the form $h = (X, V_1, \ldots, V_{va(G)})$, where $(V_1, \ldots, V_{va(G)})$ is the partition of $V(G)$, which realizes the number $va(G)$. Thus $\chi_{(\mathcal{S},\mathcal{D}_1)}(S(G)) = va(G) + 1$ for connected graphs. If G is disconnected with components $G_1, \ldots, G_n$, then the assertion follows by the facts: $\chi_{(\mathcal{S},\mathcal{D}_1)}(G) = \max\{\chi_{(\mathcal{S},\mathcal{D}_1)}(G_i) : 1 \leq i \leq n\}$ and $va(G) = \max\{va(G_i) : 1 \leq i \leq n\}$. □

The number $va(G)$ was widely studied by many researchers. In particular, there are known facts that $va(G) \leq 3$ for each planar graph G and $va(G) \leq 2$ for each bipartite or outerplanar graph G [3]. It implies the following corollaries.

Corollary 1. *If a graph G is planar, then $\chi_{(\mathcal{S},\mathcal{D}_1)}(S(G)) \leq 4$.*

Corollary 2. *If a graph G is bipartite or outerplanar, then $\chi_{(\mathcal{S},\mathcal{D}_1)}(S(G)) \leq 3$.*

In [8,18], the authors considered the complexity of the following decision problem.

GRAPH k-VERTEX ARBORICITY PROBLEM
INSTANCE: A graph G, $k \geq 2$.
QUESTION: Is $va(G) \leq k$?