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The pebbling number of squares of even cycles☆

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ABSTRACT

A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a connected graph G, denoted by $f(G)$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified vertex by a sequence of pebbling moves. In this paper, we determine the pebbling numbers of squares of even cycles.

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1. Introduction

Pebbling of graphs was first introduced by Chung [1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one of those pebbles on an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G, v)$ with the property that from every placement of $f(G, v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum of $f(G, v)$ over all the vertices of G.

There are many known results about pebbling number (see [1,5,4,6,7,2,3]). If each vertex (except v) has at most one pebble, then no pebble can be moved to v. Also, if u is of distance d from v and at most $2^d - 1$ pebbles are placed on u (and none elsewhere), then no pebble can be moved from u to v. So it is clear that $f(G) \geqslant \max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G, and D is the diameter of G. Furthermore, $f(K_n) = n$ and $f(P_n) = 2^{n-1}$ (see [1]), where K_n denotes a complete graph with n vertices and P_n denotes a path with n vertices.

Throughout this paper, G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let p be a distribution of pebbles on G. Define $p(H)$ to be the number of pebbles on a subgraph H of G and $p(v)$ to be the number of pebbles on a vertex v of G. Moreover, denote by $\widetilde{p}(H)$ and $\widetilde{p}(v)$ the number of pebbles on H and the number of pebbles on v after a specified sequence of pebbling moves, respectively. For $uv \in E(G)$, $u \xrightarrow{m} v$ refers to taking $2m$ pebbles off u and placing m pebbles on v. Denote by $\langle v_0, v_1, \ldots, v_{n-1}\rangle$ (respectively, $[v_0, v_1, \ldots, v_{n-1}]$) the path (respectively, cycle) with vertices $v_0, v_1, \ldots, v_{n-1}$ in order.

Let G be a connected graph. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the distance between u and v in G. The kth power of G, denoted by G^k, is the graph obtained from G by adding the edge uv to G whenever $2 \leqslant d_G(u, v) \leqslant k$ in G. That is, $E(G^k) = \{uv : 1 \leqslant d_G(u, v) \leqslant k\}$. Obviously, G^k is the complete graph whenever k is at least the diameter of G. We now introduce a lemma which will be used in the subsequent proofs.

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Lemma 1 ([6]). $f(P_{2k}^2) = 2^k, f(P_{2k+1}^2) = 2^k + 1$.

In [6], Pachter et al. gave the pebbling numbers of squares of paths (see Lemma 1). Naturally, we want to know the pebbling number of C_n^2. In [8], the pebbling numbers of squares of odd cycles were obtained:

(i) for $2 \leqslant n \leqslant 6, f(C_{2n+1}^2) = 2n + 1$;
(ii) for $k \geqslant 3, f(C_{4k+3}^2) = 2^{k+1} + 1$;
(iii) for $k \geqslant 4, f(C_{4k+1}^2) = \left\lceil \frac{2^{k+2}}{3} \right\rceil + 1$.

Motivated by this, we obtain the pebbling numbers of squares of even cycles in this paper.

2. Pebbling C_{2n}^2

This section studies the pebbling number of C_{2n}^2. Let $C_{2n} = [v, a_1, \ldots, a_{n-1}, y, b_{n-1}, \ldots, b_1]$. By symmetry, we may assume that v is the target vertex in C_{2n}^2 and $p(v) = 0$. First, we give the pebbling number of C_{2n}^2 for $n \leqslant 6$. See Theorems 2 and 3.

Theorem 2. *For* $2 \leqslant n \leqslant 5, f(C_{2n}^2) = 2n$.

Proof. Let $Q_A = \langle v, a_1, \ldots, a_{n-1} \rangle$ and $Q_B = \langle v, b_1, \ldots, b_{n-1} \rangle$. For $2 \leqslant n \leqslant 5$, we have $f(Q_A^2) = f(Q_B^2) = n$ by Lemma 1. Clearly, $f(C_{2n}^2) \geqslant 2n$. Now distribute $2n$ pebbles on C_{2n}^2. Without loss of generality, we assume that $p(Q_A^2) \geqslant p(Q_B^2)$. Thus $p(Q_A^2) \geqslant \left\lceil \frac{2n-p(y)}{2} \right\rceil = n - \left\lfloor \frac{p(y)}{2} \right\rfloor$. Since we can move $\left\lfloor \frac{p(y)}{2} \right\rfloor$ pebbles from y to a_{n-1}, $\widetilde{p}(Q_A^2) \geqslant p(Q_A^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor \geqslant n$. The proof is completed. □

Theorem 3. $f(C_{12}^2) = 12$.

Proof. Let $Q_A = \langle v, a_1, a_2, a_3, a_4 \rangle$ and $Q_B = \langle v, b_1, b_2, b_3, b_4 \rangle$. Moreover, let $Q_A^+ = \langle v, a_1, a_2, a_3, a_4, a_5 \rangle$ and $Q_B^+ = \langle v, b_1, b_2, b_3, b_4, b_5 \rangle$. By Lemma 1, we have $f(Q_A^2) = f(Q_B^2) = 5$ and $f((Q_A^+)^2) = f((Q_B^+)^2) = 8$. Clearly, $f(C_{12}^2) \geqslant 12$. For convenience, a_5 and b_5 are denoted by x and z, respectively. Suppose that there are 12 pebbles distributed on the vertices of C_{12}^2, i.e.,

$$p(Q_A^2) + p(Q_B^2) + p(x) + p(y) + p(z) = 12. \tag{1}$$

We first consider the case $p(x) + p(z) \geqslant 10$. It suffices to show that, after some pebbling moves, $\widetilde{p}((Q_A^+)^2) \geqslant 8$ or $\widetilde{p}((Q_B^+)^2) \geqslant 8$. Without loss of generality, we may assume that $p(x) \geqslant p(z)$. If $p(x) > \max\{p(z), 5\}$, then $p(x) + \left\lfloor \frac{p(z)}{2} \right\rfloor \geqslant 8$, and this implies $\widetilde{p}((Q_A^+)^2) \geqslant 8$. Now suppose that $p(x) = p(z) = 5$. If $p(Q_A^2) = p(Q_B^2) = 0$, then $p(y) = 2$. Moving one pebble from y to x, we have $\widetilde{p}(x) > 5$, and the previous case applies. Otherwise, suppose without loss of generality that Q_A^2 has at least one pebble; now 2 pebbles can be moved from z to x to obtain $\widetilde{p}((Q_A^+)^2) \geqslant 8$.

Next, we consider the case $p(x) + p(z) < 10$. Obviously, if $\widetilde{p}(Q_A^2) \geqslant 5$ or $\widetilde{p}(Q_B^2) \geqslant 5$, then we are done. If $\widetilde{p}(Q_A^2) < 5$ and $\widetilde{p}(Q_B^2) < 5$, then

$$p(Q_A^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor + \left\lfloor \frac{p(x) + \left\lfloor \frac{p(z)}{2} \right\rfloor}{2} \right\rfloor \leqslant 4 \tag{2}$$

and

$$p(Q_B^2) + \left\lfloor \frac{p(y)}{2} \right\rfloor + \left\lfloor \frac{p(z) + \left\lfloor \frac{p(x)}{2} \right\rfloor}{2} \right\rfloor \leqslant 4. \tag{3}$$

(2) and (3) result from moving as many pebbles as possible from x, y, z to a_4 and b_4, respectively. We see that z can contribute pebbles not only to x, but also to y. If $p(z) \geqslant 2$, then we can move pebbles from z to x and y to make at least one of $\widetilde{p}(x)$ and $\widetilde{p}(y)$ be even. So (2) can be rewritten as $p(Q_A^2) + \left\lfloor \frac{p(x)+p(y)+\left\lfloor \frac{p(z)}{2} \right\rfloor}{2} \right\rfloor \leqslant 4$ for $p(z) \geqslant 2$.

For the case $\min\{p(x), p(z)\} \geqslant 2$, we have

$$p(Q_A^2) + \left\lfloor \frac{p(x) + p(y) + \left\lfloor \frac{p(z)}{2} \right\rfloor}{2} \right\rfloor \leqslant 4 \quad \text{and} \quad p(Q_B^2) + \left\lfloor \frac{p(z) + p(y) + \left\lfloor \frac{p(x)}{2} \right\rfloor}{2} \right\rfloor \leqslant 4. \tag{4}$$

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