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Properties, isomorphisms and enumeration of 2-Quasi-Magic Sudoku grids

Siân K. Jones, Stephanie Perkins *, Paul A. Roach

Division of Mathematics and Statistics, University of Glamorgan, Pontypridd, Mid Glamorgan CF37 1DL, UK

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ABSTRACT

A Sudoku grid is a 9 × 9 Latin square further constrained to have nine non-overlapping 3 × 3 mini-grids each of which contains the values 1–9. In Δ-Quasi-Magic Sudoku a further constraint is imposed such that every row, column and diagonal in each mini-grid sums to an integer in the interval $[15-\Delta, 15+\Delta]$. The problem of proving certain (computationally known) results for $\Delta = 2$ concerning mini-grids and bands (rows of mini-grids) was posed at the British Combinatorial Conference in 2007. These proofs are presented and extensions of these provide a full combinatorial enumeration for the total number of completed 2-Quasi-Magic Sudoku grids. It is also shown that there are 40 isomorphism classes of completed 2-Quasi-Magic Sudoku grids.

1. Introduction

A Sudoku grid is a 9 × 9 Latin square further subdivided into nine non-overlapping 3 × 3 mini-grids, so that every *row*, *column* and *mini-grid* contains the values 1, . . . , 9. The 9 × 9 grid will be thought of as being subdivided into 3 *bands*, each being composed of three horizontally-consecutive mini-grids, and three *stacks*, each being composed of three vertically-consecutive mini-grids. Each 3 × 3 mini-grid possesses three sub-rows, or *tiers*, and three sub-columns, or *pillars*. A Δ-Quasi-Magic Sudoku grid has the further constraint that within each mini-grid the *tier-triples*, *pillar-triples* and *diagonal-triples* sum to any integer in the interval $[15 - \Delta, 15 + \Delta]$. This will be referred to here as the *sum constraint*. It was shown in [2] that there do not exist any Δ-Quasi-Magic Sudoku grids in which Δ is either 0 or 1.

This paper focuses on 2-Quasi-Magic Sudoku grids in which the values in each tier, pillar and diagonal of every mini-grid sum to an integer in the interval [13, 17]. A 2-Quasi-Magic Sudoku grid with some values removed is termed a *puzzle*; such a puzzle and its corresponding solution grid (which will be referred to throughout as a 2-Quasi-Magic Sudoku grid) are given in Fig. 1(a) and (b) respectively.

Proofs of certain inherent grid properties for 2-Quasi-Magic Sudoku were requested at the British Combinatorial Conference in 2007 [1]. These properties are identified, and their proofs presented, in Sections 2 and 3.

This paper is structured so that properties relating to individual mini-grids and bands of 2-Quasi-Magic Sudoku grids are presented first, in Sections 2 and 3. These properties are then extended in Section 4 to provide properties that relate to complete grids, which are subsequently used in Section 5 to show that the total number of 2-Quasi-Magic Sudoku grids is 248 832. This result was previously known through computational means [4]. The symmetry group of Sudoku was given in [6] and used to provide an enumeration of the number of non-isomorphic completed Sudoku grids. The symmetry groups of 2-Quasi-Magic Sudoku form a subset of these and a similar analysis in Section 6 shows that there are 40 isomorphism classes of completed 2-Quasi-Magic Sudoku grids. The study of 2-Quasi-Magic Sudoku is of mathematical interest due to its relationship with similar combinatorial structures. The methods of proof presented in this paper may be extendible to

* Corresponding author.
E-mail addresses: skjones@glam.ac.uk (S.K. Jones), sperkins@glam.ac.uk (S. Perkins), proach@glam.ac.uk (P.A. Roach).

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						9		
		8	1				4	
5			7	2			8	3
1			5		2	8		
	5			3			7	
		9	8		7			2
9	1			4	8			6
	4				1	3		
		3						

(a) 2-Quasi-Magic Sudoku puzzle.

7	2	4	6	8	3	9	1	5
3	6	8	1	5	9	2	4	7
5	9	1	7	2	4	6	8	3
1	7	6	5	9	2	8	3	4
8	5	2	4	3	6	1	7	9
4	3	9	8	1	7	5	6	2
9	1	5	3	4	8	7	2	6
2	4	7	9	6	1	3	5	8
6	8	3	2	7	5	4	9	1

(b) 2-Quasi-Magic Sudoku solution grid.

Fig. 1. Example 2-Quasi-Magic Sudoku puzzle and solution grid [2].

Table 1
Permutation operations generating $\mathfrak{Q}$.

Permutation operation	Order
Permutation of the values $(1, 9)(2, 8)(3, 7)(4, 6)(5)$, referred to as permutation of values.	2
Permutation of bands.	3!
Permutation of stacks.	3!
Permutation of the entire top row of any band with the entire bottom row of the same band, referred to as permutation of rows.	2^3
Permutation of the entire left column of any stack with the entire right column of the same stack, referred to as permutation of columns.	2^3
Reflection in the main diagonal *i.e.*$[Q_{a,b}]_{i,j} \mapsto [Q_{b,a}]_{j,i}$.	2

Δ-Quasi-Magic Sudoku for $\Delta \in \{3, \ldots, 9\}$, or similar combinatorial structures in which the dimension of the grid is larger, or in which the grid contains different constraints.

The following notation will be used throughout the paper. Let Q be a 2-Quasi-Magic Sudoku grid and let $Q_{a,b}$ represent the mini-grid in band a and stack b. Then the cell in tier i and pillar j of $Q_{a,b}$ is $[Q_{a,b}]_{i,j}$. The cells in mini-grids are further described as centre cells when $i = j = 2$, corner cells for $i, j \in \{1, 3\}$ and edge cells otherwise. Let the 3×3 array C represent the values in the centre of the mini-grids such that $C_{p,q} = [Q_{p,q}]_{2,2}$ for all p, q.

2. Mini-grid properties

Recall that all triples within each mini-grid sum to an integer in the interval $[13, 17]$, referred to as the sum constraint. The following immediate observation can be made.

Observation 1. *Both values from either* $\{1, 2\}$ *or* $\{8, 9\}$ *cannot occur in any triple in a mini-grid and hence none of the values* $1, 2, 8, 9$ *can occur as a mini-grid centre.*

C can therefore contain only the values: 3, 4, 5, 6 and 7.

In order to simplify the identification of properties, and enumeration, of 2-Quasi-Magic Sudoku the symmetry group, $\mathfrak{Q}$, is defined here. Applying an element of $\mathfrak{Q}$ to a 2-Quasi-Magic Sudoku grid requires the resulting grid to retain the properties of 2-Quasi-Magic Sudoku. The symmetry group, $\mathfrak{Q}$, for 2-Quasi-Magic Sudoku is formed by the permutations given in Table 1.

The number of symmetries in $\mathfrak{Q}$ is 9216 as given in Table 1. It should be noted that $\mathfrak{Q}$ also permutes the values in C. Two 2-Quasi-Magic Sudoku grids Y and Z will be considered isomorphic if $f(Y) = Z$ and $f'(Z) = Y$ where f is a symmetry operation in $\mathfrak{Q}$ and f' its inverse.

Lemma 2. *In any mini-grid* $Q_{a,b}$ *with* $[Q_{a,b}]_{2,2} = x$ *for* $x \in \{3, 7\}$, *the values from* $\{1, x, 9\}$ *appear only as either the second tier or second pillar of the mini-grid.*

This can be proved using a simple contradiction argument on the placement of the value 1 (or 9).

Lemma 3. *If* $[Q_{a,b}]_{2,2} = 3$ *and the second tier of* $Q_{a,b}$ *contains the values from* $\{1, 3, 9\}$, *then the first and third tiers each contain all of the values from one of* $\{2, 6, 7\}$ *or* $\{4, 5, 8\}$ *and are distinct, and all the values from* $\{2, 3, 8\}$ *occur in a diagonal-triple. The same is true for pillars if the values from* $\{1, 3, 9\}$ *are in the second pillar of* $Q_{a,b}$.

Proof. Follows immediately from Lemma 2 and the sum constraint. □