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Kernels by monochromatic paths in digraphs with covering number 2[☆]

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ABSTRACT

We call the digraph D an k-colored digraph if the arcs of D are colored with k colors. A subdigraph H of D is called monochromatic if all of its arcs are colored alike. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$, there is no monochromatic directed path between them, and (ii) for every vertex $x \in (V(D) \setminus N)$, there is a vertex $y \in N$ such that there is an xy-monochromatic directed path. In this paper, we prove that if D is an k-colored digraph that can be partitioned into two vertex-disjoint transitive tournaments such that every directed cycle of length 3, 4 or 5 is monochromatic, then D has a kernel by monochromatic paths. This result gives a positive answer (for this family of digraphs) of the following question, which has motivated many results in monochromatic kernel theory: *Is there a natural number l such that if a digraph D is k-colored so that every directed cycle of length at most l is monochromatic, then D has a kernel by monochromatic paths?*

1. Introduction

Let D be a digraph. We denote by $V(D)$ and $A(D)$ the sets of vertices and the set of arcs of D, respectively. Let $v \in V(D)$. We denote by $N^+(v)$ and $N^-(v)$ the out- and in-neighborhood of v in D, respectively. We define $\delta^+(w) = |N^+(w)|$ and $\delta^-(w) = |N^-(w)|$. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S. For two disjoint subsets U, V of $V(D)$, we denote by $(U, V) = \{uv \in A(D) : u \in U, v \in V\}$ and $[U, V] = (U, V) \cup (V, U)$. An UV-arc is an arc from (U, V) if $U = \{u\}$ (resp. $V = \{v\}$), we denote the UV-arc by uV-arc (resp. Uv-arc). We call the digraph D an *k-colored* digraph if the arcs of D are colored with k colors. The digraph D will be an k-colored digraph and all the paths, cycles and walks considered in this paper will be directed paths, cycles or walks. If $W = (x_0, x_1, \ldots, x_n)$ is a walk, the *length* of W is n. The length of a walk W is denoted by $l(W)$. The path $(u_0, u_1, \ldots, u_n)$ will be called an UV-path whenever $u_0 \in U$ and $u_n \in V$. A *tournament* is a digraph T such that there is exactly one arc between any two vertices of T. An acyclic tournament is called a *transitive tournament*. A vertex $v \in V(T)$ is called a *sink* if $N^-(v) = V(D) \setminus v$. A subdigraph H of a k-colored digraph D is called *monochromatic* if all of its arcs are colored alike. Let $N \subseteq V(D)$. Then, N is said to be *m-independent* if there is no monochromatic directed path between any pair of vertices of the set N, N is a *m-absorbent* (or m-dominant) if for every vertex $x \in (V(D) \setminus N)$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path and finally, N is a *m-kernel* (kernel by monochromatic paths) if it satisfies the following two conditions: (i) N is m-independent and (ii) N is m-absorbent. For general concepts, we refer the reader to [1,2,7].

The topic of domination in graphs has been widely studied by many authors. A very complete study of this topic is presented in [19,20]. A special class of domination is the domination in digraphs, and it is defined as follows. In a digraph D,

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a set of vertices $S \subseteq V(D)$ dominates whenever for every $w \in (V(D) \setminus S)$ there exists a wS-arc in D. Dominating independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (see for instance [21,22,8,9,26]) and they have been studied by several authors; interesting surveys of kernels in digraphs can be found in [6,9]. The concepts of m-domination, m-independence and m-kernel in edge-colored digraphs are generalization of those of domination, independence and kernel in digraphs. The study of the existence of m-kernels in edge-colored digraphs starts with the theorem of Sands, Sauer and Woodrow, proved in [25], which asserts that every two-colored digraph possesses an m-kernel. In the same work, the authors proposed the following question: let D be an k-colored tournament such that every directed cycle of length 3 is quasi-monochromatic (a subdigraph H of an k-colored digraph D is said to be *quasi-monochromatic* if, with at most one exception, all of its arcs are colored alike) must D have a m-kernel? Minggan [24] proved that if D is an k-colored tournament such that every directed cycle of length 3 and every transitive tournament of order 3 is quasi-monochromatic, then D has a m-kernel. He also proved that this result is best possible for $m \geq 5$. In [15], it was proved that the result is best possible for $m \geq 4$. The question for $m = 3$ is still open: *Does every* 3*-colored tournament such that every directed cycle of length* 3 *is quasi-monochromatic have a m-kernel*? Sufficient conditions for the existence of m-kernels in edge-colored digraphs have been obtained mainly in tournaments and generalized tournaments, and ask for the monochromaticity or quasi-monochromaticity of small digraphs (due to the difficulty of the problem) in several papers (see [10,11,15,16,18,24]).Other interesting results can be found in [27,28]. Another question which has motivated many results in m-kernel theory is the following (proposed in the abstract): *Given a digraph D is there an integer k such that if every directed cycle of length at most k is monochromatic (resp. quasi-monochromatic), then D has a m-kernel*? In [11] (resp. in [16]) it was proved that if D is an k-colored tournament (resp. bipartite tournament) such that every directed cycle of length 3 (resp. every directed cycle of length 4) is monochromatic, then D has a m-kernel. Later the following generalization of both results was proved in [17]: if D is an k-colored k-partite tournament, such that every directed cycle of length 3 and every directed cycle of length 4 is monochromatic, then D has a m-kernel. In [18] were considered quasi-monochromatic cycles, the authors proved that if D is an k-colored tournament such that for some k every directed cycle of length k is quasi-monochromatic and every directed cycle of length less than k is not polychromatic (a subdigraph H of D is called *polychromatic* whenever it is colored with at least three colors), then D has a m-kernel. In [13] this result was extended for nearly complete digraphs. The *covering number* of a digraph D is the minimum number of transitive tournaments of D that partition $V(D)$. Digraphs with a small covering number are a nice class of nearly tournament digraphs. The existence of kernels in digraphs with a covering number at most 3 has been studied by several authors, in particular by Berge [3], Maffray [23] and others [4,5,12,14].

In this paper, we study the existence of m-kernel in edge-colored digraphs with covering number 2, asking for the monochromaticity of small directed cycles. We prove that if D is an k-colored digraph with covering number 2 such that every directed cycle of length 3, 4 or 5 is monochromatic, then D has a m-kernel.

2. Structural properties

We consider the family $\mathfrak{D}$ of digraphs D with covering number 2. Since D has covering number 2, there exists a non-trivial partition of $V(D)$ into two sets U, V such that $D[U]$, $D[V]$ are transitive tournaments. Throughout this paper, the non-trivial partition of the vertex set into U, V is such that $D[U]$, $D[V]$ are transitive tournaments. Let T be a transitive tournament of order n. Throughout this paper, $(v_n, v_{n-1}, \ldots, v_1)$ will denote the Hamiltonian path in T. Thus for any $1 \leq i \leq n$, the vertex v_i is the sink of $T \setminus \{v_1, v_2, \ldots, v_{i-1}\}$, in particular, v_1 is the sink of T. When $P = (u_0, u_1, \ldots, u_k)$ is a path, we will denote by (u_i, P, u_j), for $0 \leq i < j \leq k$, the u_i, u_j-path contained in P. Let u_iu_{i+1} and u_ju_{j+1} be two distinct arcs on P. We say that the arc u_iu_{i+1} *precedes* (resp. *follows*) the arc u_ju_{j+1} on the path P, if $i < j$ (resp. $j < i$).

Throughout this paper, the vertex z will be **fixed and arbitrary**.

First, we prove some structural properties of the wz-paths of minimum length with $w \in \{u_1, v_1\}$ in digraphs of the family $\mathfrak{D}$. Next, we extend these properties for wz-paths of minimum length with $w \in \{u_1, v_1\}$ in k-colored digraphs of the family $\mathfrak{D}$ with every directed cycle of length 3, 4 or 5 monochromatic.

Let $u_mv_n, v_ku_l \in [U, V]$. We say that u_mv_n, v_ku_l are *crossing arcs* if $u_m, u_l \in V(D)$, $v_n, v_k \in V(D)$, and $m \leq l, k \leq n$, except when $n = k$ and $m = l$ (see Fig. 1). Let $u_mv_n \in (U, V)$. If $xy, u_mv_n \in [U, V]$ are crossing arcs, then clearly $xy \in (V, U)$.

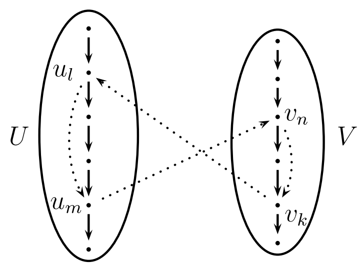

Fig. 1. u_mv_n and v_ku_l are crossing arcs.