Discrete Mathematics 312 (2012) 3461–3466

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Irregular labelings of circulant graphs

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ARTICLE INFO

Article history:
Received 3 January 2012
Accepted 18 June 2012
Available online 9 July 2012

Keywords:
Irregularity strength
Total vertex irregularity strength
Graph weighting
Graph labeling
Circulant graph

ABSTRACT

We investigate the *irregularity strength* (s(G)) and *total vertex irregularity strength* (tvs(G)) of circulant graphs $Ci_n(1, 2, \ldots, k)$ and prove that $\text{tvs}(Ci_n(1, 2, \ldots, k)) = \lceil \frac{n+2k}{2k+1} \rceil$, while $\text{s}(Ci_n(1, 2, \ldots, k)) = \lceil \frac{n+2k-1}{2k} \rceil$ except if either $n = 2k + 1$ or if k is odd and $n \equiv 2k + 1 \pmod{4k}$, then $\text{s}(Ci_n(1, 2, \ldots, k)) = \lceil \frac{n+2k-1}{2k} \rceil + 1$.

1. Introduction

Let us consider a simple undirected graph $G = (V(G), E(G))$ without loops, without isolated edges and with at most one isolated vertex. We assign a label $w(e)$ (called also weight), being a positive integer, to every edge $e \in E(G)$. For every vertex $v \in V(G)$ we define its *weighted degree* as

$$wd(v) = \sum_{e \ni v} w(e).$$

We call a weighting w *irregular* if for each pair of vertices, their weighted degrees are distinct. In [8] the authors defined the graph parameter s(G) called the *irregularity strength* of G being the smallest integer s such that there exists an irregular weighting of G with integers $\{1, 2, \ldots, s\}$. The value of s(G) is known only for some special classes of graphs, e.g. complete graphs [8], graphs with the components being paths and cycles [2,11], or some families of trees [3,14].

The lower bound on the s(G) is given by the inequality

$$\text{s}(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}, \tag{1}$$

where n_i denotes the number of vertices of degree i. In the case of d-regular graphs it reduces to

$$\text{s}(G) \geq \frac{n + d - 1}{d}. \tag{2}$$

The conjecture stated in [8] says that the value of s(G) is for every graph equal to the above lower bound plus some constant not depending on G. The first upper bounds including the vertex degrees in the denominator were given in [9]

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2012.06.017

(cn/δ with relatively large values of c, depending on the relation between n, δ and Δ), then improved (slight reduction of c) in [12,13]. The best upper bounds known so far can be found in [10]. Namely, the authors have proved that

$$s(G) \le \left\lceil \frac{6n}{\delta} \right\rceil. \quad (3)$$

The following variant of irregularity strength that allows the vertices to be labeled as well was introduced in [6]. Now the weighted degree is defined as

$$wd(v) = \sum_{e \ni v} w(e) + w(v).$$

The corresponding graph parameter, tvs(G), is called *total vertex irregularity strength*. The authors of [6] gave the following lower and upper bounds:

$$\left\lceil \frac{n + \delta(G)}{\Delta(G) + 1} \right\rceil \le \mathrm{tvs}(G) \le n + \Delta(G) - 2\delta(G) + 1. \quad (4)$$

In the case of d-regular graphs this reduces to

$$\left\lceil \frac{n + d}{d + 1} \right\rceil \le \mathrm{tvs}(G) \le n - d + 1. \quad (5)$$

The exact values of tvs(G) are known only for a few families of graphs, e.g. complete graphs, paths and cycles [6] or forests without vertices of degree 2 [5]. The best upper bound on tvs(G) is given in [4]:

$$\mathrm{tvs}(G) \le \left\lceil \frac{3n}{\delta} \right\rceil + 1. \quad (6)$$

Let us consider circulant graphs defined as follows (see e.g. [7]).

Definition 1.1. Let n and $s_1, s_2, \dots, s_k$ be integers, with $1 \le s_1 < \cdots < s_k \le n/2$. The circulant graph $G = Ci_n(s_1, \dots, s_k)$ of order n is a graph with vertex set $V(G) = \{0, 1, \dots, n-1\}$ and edge set $E(G) = \{(x, x \pm s_i \bmod n), x \in V(G), 1 \le i \le k\}$.

Note that $Ci_n(s_1, \dots, s_k)$ is $2k$-regular. The main result given in [7] says that in the case $k = 2$ and $s_1 = 1$,

$$s(Ci_n(1, s_2)) = \left\lceil \frac{n + 3}{4} \right\rceil \quad (7)$$

if only $s_2 \ge 2$ and $n \ge 4s_2 + 1$. Observe that in this case the value s(G) is equal to the lower bound given by (2).

In [1] the authors gave the exact value of total vertex irregularity strength of the graphs $Ci_n(1, 2)$:

$$\mathrm{tvs}(Ci_n(1, 2)) = \left\lceil \frac{n + 4}{5} \right\rceil. \quad (8)$$

In this paper we consider a more general case of circulant graphs, $Ci_n(1, 2, \dots, k)$, i.e. the k-th powers of cycles C_n^k. We prove the following two theorems.

Theorem 1.2. *If $k \ge 2$ and $n \ge 2k + 1$, then*

$$\mathrm{tvs}(C_n^k) = \left\lceil \frac{n + 2k}{2k + 1} \right\rceil.$$

Theorem 1.3. *If $k \ge 2$ and $n \ge 2k + 1$, then*

$$s(C_n^k) = \begin{cases} \left\lceil \frac{n + 2k - 1}{2k} \right\rceil + 1, & \text{if } n = 2k + 1 \text{ or if } n \equiv 2k + 1 \pmod{4k} \text{ and } k \text{ is odd,} \\ \left\lceil \frac{n + 2k - 1}{2k} \right\rceil, & \text{otherwise.} \end{cases}$$

2. Proof of Theorem 1.2

The lower bound on tvs(G) is given by (5), we now present an upper bound. For simplicity write $\lceil \frac{n+2k}{2k+1} \rceil = \lceil \frac{n-1}{2k+1} \rceil + 1 = s + 1$. Further, observe that because C_n^k is a regular graph it is enough to find an irregular weighting with weights $\{0, 1, 2, \dots, s\}$ (to complete the final weighting just add 1 to every edge and vertex label). Finally, throughout the proof if