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On the facial Thue choice index of plane graphs

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ABSTRACT

Let G be a plane graph, and let φ be a colouring of its edges. The edge colouring φ of G is called facial non-repetitive if for no sequence $r_1, r_2, \ldots, r_{2n}$, $n \geq 1$, of consecutive edge colours of any facial path we have $r_i = r_{n+i}$ for all $i = 1, 2, \ldots, n$. Assume that each edge e of a plane graph G is endowed with a list $L(e)$ of colours, one of which has to be chosen to colour e. The smallest integer k such that for every list assignment with minimum list length at least k there exists a facial non-repetitive edge colouring of G with colours from the associated lists is the facial Thue choice index of G, and it is denoted by $\pi_{fl}(G)$. In this article we show that $\pi'_{fl}(G) \leq 291$ for arbitrary plane graphs G. Moreover, we give some better bounds for special classes of plane graphs.

1. Introduction

A finite sequence $r_1, r_2, \ldots, r_{2n}$ of symbols is a *repetition* if $r_i = r_{n+i}$ for all $i = 1, 2, \ldots, n$. On the other hand, a finite or infinite sequence $a_1, a_2, \ldots$ is called *non-repetitive* if it does not contain any subsequence of its consecutive terms which is a repetition. In 1906, the Norwegian mathematician and number theoretician Axel Thue started the systematic study of word structure. In his seminal paper [13], he showed that there are arbitrarily long non-repetitive sequences over three symbols. Since then, Thue's non-repetitive sequences have repetitively occurred in mathematics, and various questions concerning non-repetitive colourings of graphs have been formulated (see, e.g., [6]). For the case of vertex colourings, probably the most intriguing open problem is whether there is a finite bound for the number of colours needed to colour the vertices of any planar graph G in such a way that the colours of no path of G form a repetition. One way to relax the requirements in this problem is to consider not all paths of a plane graph but only paths belonging to the boundary walk of a face.

We consider the case of edge colourings. Let G be a simple graph, and let φ be a colouring of its edges. We say that φ is *non-repetitive* (or *square-free*) if for any simple path on edges $e_1, e_2, \ldots, e_{2n}$ in G the associated sequence of colours $\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_{2n})$ is not a repetition. The minimal number of colours needed in such a colouring is the *Thue chromatic index* of G, and it is denoted by $\pi'(G)$. If G is a plane graph, a *facial non-repetitive edge colouring* of G is an edge colouring such that any *facial path* (i.e., a path of consecutive edges on the boundary walk of a face) is coloured non-repetitively. The minimum number of colours needed in such a colouring is the *facial Thue chromatic index* of G, and it is denoted by $\pi'_f(G)$. The concept of facial non-repetitive edge colourings was studied in [9,10,12]. It was shown that $\pi'_f(G) \leq 8$ for every plane graph G, while better bounds were given by the authors for several subclasses of plane graphs. Facial non-repetitive vertex colourings were considered in [1]. It was shown there that, for any plane graph G, there is a facial non-repetitive vertex colouring using at most 24 colours.

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Later, non-repetitive list colourings of graphs also drew attention (see, e.g., [4,7,8]). Using probabilistic arguments, Czerwiński and Grytczuk [4] proved that for every graph G with maximum degree $\Delta(G)$ there exists a vertex colouring from lists of size at least $16\Delta(G)^{2-\frac{1}{k}}$ with no repetitive path on at most $2k$ vertices. In the same paper, they conjectured that every path has a non-repetitive colouring from arbitrary lists of size 3. Moreover, recently, Grytczuk et al. [8], using the left-handed Local Lemma, proved that every path has a non-repetitive vertex colouring from arbitrary lists of size 4. A more constructive proof of the same theorem is given in [7].

For the case of list edge colourings, let the *Thue choice index* $\pi'_{ch}(G)$ of a graph G denote the smallest integer k such that, for any list assignment $L : E(G) \to 2^{\mathbb{N}}$ with minimum list length at least k, there is a colouring of the edges from the assigned lists such that the sequence of edge colours of no path in G forms a repetition. Since the line graph of a path P is a path too, the vertex colouring result for paths implies the following.

Theorem 1 ([8,7]). *Every path P_n satisfies $\pi'_{ch}(P_n) \leq 4$.*

We consider the relaxation of the edge colouring problem, where only the facial paths of a plane graph G are required to be coloured non-repetitively. A *facial non-repetitive list edge colouring* of a graph G with list assignment $L : E(G) \to 2^{\mathbb{N}}$ is a facial non-repetitive edge colouring φ_{fl} of G such that the colour $\varphi_{fl}(e)$ of each edge e is chosen from the list $L(e)$. If such a colouring exists for any list assignment L with minimum list length at least k, we call *G facial non-repetitively k-edge choosable*. The minimum integer k such that G is facial non-repetitively k-edge choosable is the *facial Thue choice index* of G, and we denote it by $\pi'_{fl}(G)$. Since any non-repetitive edge colouring of a plane graph G is also a facial non-repetitive edge colouring, we immediately obtain $\pi'_{fl}(G) \leq \pi'_{ch}(G)$. Inspired by the result of Czerwiński and Grytczuk [4] on the Thue choice index of paths, we show some results concerning facial non-repetitive list edge colourings of plane graphs.

2. A general bound for plane graphs

In order to prove our main result, we will need a special version of the local lemma. It is the asymmetric version of the Lopsided Lovász Local Lemma introduced in [5]. Here, all bad events may be pairwise dependent, but intuitively the influence of some events on a given event is only limited. So for each bad event we can partition the other events into a set of events with high influence and a set of events with limited influence. For the sake of completeness, we include a proof of the lemma, which basically follows the standard lines for the proof of the asymmetric local lemma.

Lemma 1. *Let $A_1, \ldots, A_n$ be events, and for each event A_i let D_i and C_i denote subsets of $\{1, \ldots, n\}$ such that $D_i \cup C_i = \{1, \ldots, n\} \setminus \{i\}$ and $C_i \cap D_i = \emptyset$. Moreover, let $x_1, \ldots, x_n$ be numbers in $[0, 1]$ such that*

$$\Pr\left(A_i \mid \bigcap_{j\in C} \overline{A}_j\right) < x_i \cdot \prod_{k\in D_i}(1 - x_k)$$

for all $i \in \{1, \ldots, n\}$ and any $C \subseteq C_i$. Then the probability $\Pr(\bigcap_{i=1}^{n} \overline{A}_i) > 0$.

Proof. We will prove more than that, namely that $\Pr(\bigcap_{i=1}^{n} \overline{A}_i) > \prod_{i=1}^{n}(1-x_i)$. We will first show that for any $i \in \{1, \ldots, n\}$ and any set $S \subseteq \{1, \ldots, n\} \setminus \{i\}$ we have

$$\Pr\left(A_i \mid \bigcap_{j\in S} \overline{A}_j\right) < x_i. \tag{1}$$

We prove (1) by induction on $|S|$. If $S = \emptyset$, then we have $S \subseteq C_i$; hence, it follows directly from the assumption of the lemma that $\Pr(A_i \mid \bigcap_{j\in S} \overline{A}_j) < x_i \cdot \prod_{k\in D_i}(1 - x_k) \leq x_i$.

Now, assume that (1) is true for all $i \in \{1, \ldots, n\}$ and all S' with $|S'| < m$ and $i \notin S'$, and consider a fixed $i \in \{1, \ldots, n\}$ and $S \subseteq \{1, \ldots, n\} \setminus \{i\}$ with $|S| = m$. We define $S_1 := S \cap D_i$ and $S_2 := S \cap C_i$. If $S_1 = \emptyset$, then, using the assumption of the lemma for $C = S = S_2$, we obtain $\Pr(A_i \mid \bigcap_{j\in S} \overline{A}_j) < x_i \cdot \prod_{k\in D_i}(1 - x_k) \leq x_i$. Hence, we may assume that $S_1 \neq \emptyset$.

$$\Pr\left(A_i \mid \bigcap_{j\in S} \overline{A}_j\right) = \Pr\left(A_i \mid \bigcap_{j\in S_1} \overline{A}_j \cap \bigcap_{j\in S_2} \overline{A}_j\right) = \frac{\Pr\left(A_i \cap \bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j\in S_2} \overline{A}_j\right)}{\Pr\left(\bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j\in S_2} \overline{A}_j\right)}. \tag{2}$$