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Biased orientation games

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ABSTRACT

We study biased *orientation games*, in which the board is the complete graph K_n, and OMaker (oriented maker) and OBreaker (oriented breaker) take turns in directing previously undirected edges of K_n. At the end of the game, the obtained graph is a tournament. OMaker wins if the tournament has some property $\mathcal{P}$ and OBreaker wins otherwise.

We provide bounds on the bias that is required for OMaker's win and for OBreaker's win in three different games. In the first game OMaker wins if the obtained tournament has a cycle. The second game is Hamiltonicity, where OMaker wins if the obtained tournament contains a Hamilton cycle. Finally, we consider the H-creation game, where OMaker wins if the obtained tournament has a copy of some fixed digraph H.

1. Introduction

In this work, we study *orientation games*. The board consists of the edges of the complete graph K_n. In the $(p : q)$ game the two players, called *OMaker* and *OBreaker*, take turns in orienting (or directing) previously undirected edges. OMaker starts the game, and at each round OMaker directs at most p edges and then OBreaker directs at most q edges (usually, we consider the case where $p = 1$ and q is large). Both players have to direct at least one edge at each round. The game ends when all the edges are oriented, and then we obtain a *tournament*. OMaker then wins if the tournament has some fixed property $\mathcal{P}$, and OBreaker wins otherwise. Here we focus on the $1 : b$ game, which is referred to as the b-biased game. Since at each round each player has to orient at least one edge, the number of rounds is clearly bounded. It is easy to verify that the game is *bias monotone*, meaning that increasing b can only help OBreaker. Hence, every property $\mathcal{P}$ admits some threshold $t(n, \mathcal{P})$ so that OMaker wins the b-biased game if $b < t(n, \mathcal{P})$ and OBreaker wins the b-biased game if $b \geq t(n, \mathcal{P})$. We stress that in all these games OMaker wins if the obtained tournament has the desired property, no matter who directs each edge of a winning directed subgraph.

This game is an alteration of the well studied classical Maker–Breaker game, which is defined by a hypergraph $(X, \mathcal{F})$ and bias $(p : q)$. In that game, at each round Maker claims p elements of X, and Breaker claims q elements of X. Maker wins if by the end of the game he claimed all the elements of some hyperedge $A \in \mathcal{F}$, and Breaker wins otherwise. Usually, a typical problem goes as follows. Given a game hypergraph $H = (X, \mathcal{F})$, determine or estimate the threshold function t_H such that if $b > t_H$ then Breaker wins in a $(1 : b)$ game, and if $b \leq t_H$ then Maker wins in a $(1 : b)$ game. There has been a long line of research that studies the bias threshold of various games (see, e.g., [3,5,9,11–13] and their references).

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Here we study orientation games for the following three properties.

Creating a cycle. OMaker wins if the obtained tournament contains a cycle, and OBreaker wins otherwise. It is well known that a tournament contains a cycle if and only if it contains a cyclic triangle (cycle of length 3). This is a problem which has already been studied by Alon (unpublished result) and by Bollobás and Szabó [7], and here we improve their results.

Creating a Hamilton cycle. Here OMaker wins if the final tournament contains a Hamilton cycle, and OBreaker wins otherwise. Recently, the second author [12] solved a long standing question and provided tight bounds on the bias threshold for Maker's win in the classical Maker–Breaker Hamiltonicity game. We use a variant of his approach, together with a new application of the Gebauer–Szabó method [11] and give tight bounds in our case as well.

Creating a copy of H. Here we are given a fixed digraph H. OMaker wins if the obtained tournament contains a copy of H, and OBreaker wins otherwise. We provide both upper and lower bounds, and give some nearly tight bounds for specific cases. We conjecture that the correct threshold is closely related to the size of the *minimum feedback arc set* of H, and provide some results that support this conjecture.

Our first theorem considers the cycle creation game. It is easy to observe that if $b \geq n-2$ then OBreaker has a winning strategy (for completeness, we give a detailed proof in Section 3). Bollobás and Szabó [7] proved that if $b = (2-\sqrt{3})n$, then OMaker wins the game, and conjectured that the correct threshold is $b = n-2$.

In this work we provide a simple argument that improves their result.

Theorem 1 (*The Cycle Game*)**.** *For every $b \leq n/2 - 2$, OMaker has a strategy guaranteeing a cycle in the b-biased orientation game.*

The second game we consider is the Hamiltonicity game, where OMaker wins if and only if the obtained tournament contains a Hamilton cycle. Here we apply techniques from [9,11,12] to get tight bounds on the bias threshold for a win of OBreaker.

Theorem 2 (*The Hamiltonicity Game*)**.**

(i) *If $b \geq \frac{n(1+o(1))}{\ln n}$, OBreaker has a strategy to guarantee that in the b-biased orientation game the obtained tournament has a vertex of in-degree* 0*, and in particular to win the Hamiltonicity game.*

(ii) *If $b \leq \frac{n(1+o(1))}{\ln n}$, OMaker has a strategy guaranteeing a Hamilton cycle in the b-biased orientation game.*

In the H-creation game we have some partial results. We conjecture that the bias that guarantees OMaker's win depends on the minimum feedback arc set of H, and support this result for graphs with a small feedback arc set. We will present and discuss corresponding notions and results in Section 5.

2. Preliminaries

Let K_n be the complete graph on n vertices, a tournament is an orientation of K_n. A directed graph is called *oriented* if it contains neither loops nor cycles of length 2. Every oriented graph is a subgraph of a tournament. A directed graph is *strongly connected* if for every two vertices u, v there is a directed path from u to v and a directed path from v to u. All directed graphs we consider here are *oriented*, i.e., do not have parallel or opposite edges.

All log signs are in base 2, except ln signs which are in the natural base.

The well known results of Erdős and Selfridge [10] and Beck [4] give a sufficient condition for Breaker's win in biased Maker–Breaker games.

Theorem 3. *Suppose that Maker and Breaker play a $(p : q)$-game on a hypergraph $H = (V, \mathcal{F})$, with Maker starting. If*

$$\sum_{A\in\mathcal{F}} (q+1)^{-\frac{|A|}{p}} < \frac{1}{q+1},$$

then Breaker has a winning strategy.

An orientation game is defined by a series of moves by OMaker and OBreaker. In round t, OMaker orients $1 \leq m_t \leq p = p(n)$ edges (usually in our settings $p = 1$) and OBreaker orients $1 \leq b_t \leq q = q(n)$ edges. The game ends when all the edges are oriented, so the obtained graph is a tournament. OMaker wins if the tournament has some predetermined property $\mathcal{P}$, otherwise OBreaker wins.

We denote by H_t the graph containing the edges oriented after t rounds. Clearly, this graph has at most $(p+q)\cdot t$ edges.

Given a directed graph $G = (V, E)$, we write $(u, v) \in E$ if there is an edge from u to v. Given a set $A \subseteq V$, we let

$$N^+(A) = \{u \in V \setminus A : \exists v \in A, (v, u) \in E\},$$

and

$$N^-(A) = \{u \in V \setminus A : \exists v \in A, (u, v) \in E\}.$$

A tournament T on n vertices is *transitive* if there is a bijection $\sigma : V(T) \to [n]$ such that for every edge $(u, v) \in E(T)$, $\sigma(u) < \sigma(v)$. A tournament $T = (V, E)$ is k-colorable if there is a partition of V into k sets $V_1, \ldots, V_k$ such that the induced tournament on each V_i is transitive. Thus, a transitive tournament is 1-colorable.