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Facial parity edge colouring of plane pseudographs

Július Czap [a], Stanislav Jendroľ [b,*], František Kardoš [b], Roman Soták [b]

[a] Department of Applied Mathematics and Business Informatics, Faculty of Economics, Technical University of Košice, Němcovej 32, 040 01 Košice, Slovakia
[b] Institute of Mathematics, P.J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

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ABSTRACT

A *facial parity edge colouring* of a connected bridgeless plane graph is such an edge colouring in which no two face-adjacent edges receive the same colour and, in addition, for each face f and each colour c, either no edge or an odd number of edges incident with f is coloured with c. Let $\chi'_p(G)$ denote the minimum number of colours used in such a colouring of G. In this paper we prove that $\chi'_p(G) \leq 20$ for any 2-edge-connected plane graph G. In the case when G is a 3-edge-connected plane graph the upper bound for this parameter is 12. For G being 4-edge-connected plane graph we have $\chi'_p(G) \leq 9$. On the other hand we prove that some bridgeless plane graphs require at least 10 colours for such a colouring.

1. Introduction

The famous Four Colour Problem has served as a motivation for many equivalent colouring problems, see e.g. the book of Saaty and Kainen [15]. The Four Colour Problem was solved in 1976 by Appel and Haken [1] (see also [14] for another proof) and the result is presently known as the Four Colour Theorem (4CT). From the 4CT the following result follows, see [15].

Theorem 1. *The edges of a plane triangulation can be coloured with* 3 *colours so that the edges bounding every face are coloured distinctly.*

In 1965, Vizing [17] proved that simple planar graphs with maximum degree at least eight have the edge chromatic number equal to their maximum degree. He conjectured the same if the maximum degree is either seven or six. The first part of this conjecture was proved by Sanders and Zhao in 2001, see [16]. Note that (also by Vizing) every graph with maximum degree Δ has the edge chromatic number equal to Δ or $\Delta + 1$. These results of Sanders and Zhao and of Vizing can be reformulated in a sense of Theorem 1 in the following way.

Theorem 2. *Let G be a* 3*-edge-connected plane graph with maximum face size $\Delta^* \geq 7$. Then the edges of G can be coloured with Δ^* colours in such a way that the edges bounding every face of G are coloured distinctly.*

On the other hand, in 1997 Pyber [13] has shown that the edges of any simple graph can be coloured with at most 4 colours so that all the edges from the same colour class induce a graph with all vertices having odd degree. Mátrai [11] constructed an infinite sequence of finite simple graphs which require 4 colours in any such colouring. Pyber's result can be stated for plane graphs as follows.

* Corresponding author.
E-mail addresses: julius.czap@tuke.sk (J. Czap), stanislav.jendrol@upjs.sk (S. Jendroľ), frantisek.kardos@upjs.sk (F. Kardoš), roman.sotak@upjs.sk (R. Soták).

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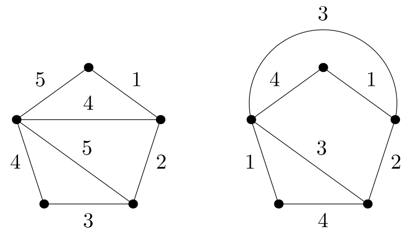

Fig. 1. Two embeddings of the same graph with different facial parity chromatic indices.

Theorem 3. *Let G be a* 3*-edge-connected plane graph. Then the edges of G can be coloured with at most* 4 *colours so that for any colour c and any face f of G either no edge or an odd number of edges on the boundary of f is coloured with colour c.*

Recently Bunde, Milans, West, and Wu [4,5] have introduced a *strong parity edge colouring* of graphs. It is an edge colouring of a graph G such that each open walk in G uses at least one colour an odd number of times. Let $p(G)$ be the minimum number of colours in a strong parity edge colouring of a graph G. The exact value of $p(K_n)$ for complete graphs is determined in [4]. They also mention that computing $p(G)$ is NP-hard even when G is a tree.

We say that an edge colouring of a plane graph G is *facially proper* if no two face-adjacent edges of G receive the same colour. (Two edges are *face-adjacent* if they are consecutive edges of a facial walk of some face f of G.) Note that colourings in Theorems 1 and 2 are facially proper, but the colouring in Pyber's Theorem 3 need not be facially proper.

Motivated by the parity edge colouring concept introduced by Bunde et al. [5] and the above mentioned theorems, a *facial parity edge colouring* of a plane graph G was defined in [7] as a facially proper edge colouring with the following property: for each colour c and each face f of G either no edge or an odd number of edges incident with f is coloured with colour c. The problem is to determine for a given bridgeless plane graph G the minimum possible number of colours, $\chi'_p(G)$, in such a colouring of G. The number $\chi'_p(G)$ is called the *facial parity chromatic index* of G.

Note that the facial parity chromatic index depends on the embedding of the graph. For example, the graph depicted in Fig. 1 has different facial parity chromatic indices depending on its embedding. With the embedding on the left, its facial parity chromatic index is 5; whereas with the embedding on the right, its facial parity chromatic index is 4.

The vertex version of this problem (*parity vertex colouring*) was introduced in [8]. The authors proved that every 2-connected plane graph G admits a proper vertex colouring with at most 118 colours such that for each face f and each colour c, either no vertex or an odd number of vertices incident with f is coloured with c. The constant 118 has been recently improved to 97 by Kaiser et al. [10]. Czap [6] proved that every 2-connected outerplane graph has a parity vertex colouring with at most 12 colours.

In this paper we prove that each connected bridgeless plane graph has a facial parity edge colouring using at most 20 colours, which improves the bound 92 published in [7]. The facial parity chromatic index is at most 12 for any 3-edge-connected plane graph. In the case when a plane graph is 4-edge-connected the upper bound is at most 9 for this parameter. We also present graphs which require 10 colours for such a colouring.

Throughout the paper, we mostly use the terminology from a recent book of Bondy and Murty [2]. All graphs considered are allowed to contain loops and multiple edges, unless stated otherwise.

2. Results

2.1. 2-edge-connected plane graphs

Let φ be a facial parity edge colouring of a bridgeless plane graph G. Observe that in the dual graph G^*, the edges of G in each colour class correspond to a factor of G^* with the degrees of all the vertices odd or zero, i.e. it is an *odd subgraph*. Moreover, since φ is a facially proper edge colouring in G, it induces a facially proper edge colouring in G^* as well.

We say that an edge colouring of a plane graph is *odd*, if each colour class induces an odd subgraph.

Observation 1. *Let G be a plane graph. Then $\chi'_p(G) \leq k$ if and only if the dual graph G^* has a facially proper odd edge colouring using at most k colours.*

This observation will play a major role in proofs below. Instead of facial parity edge colouring of G we shall investigate facially proper odd edge colouring of G^*.

Let us recall the result of Pyber in its original form.

Theorem 4 (*Pyber [13]*)**.** *The edge set of any simple graph H can be covered by at most* 4 *edge-disjoint odd subgraphs. Moreover, if H has an even number of vertices, then it can be covered by at most* 3 *edge-disjoint odd subgraphs.*