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Doyen–Wilson theorem for perfect hexagon triple systems

Hua Gao, Jinhua Wang *

School of Sciences, Nantong University, Nantong 226007, PR China

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ABSTRACT

The graph consisting of the three 3-cycles (or triples) (a, b, c), (c, d, e), and (e, f, a), where a, b, c, d, e and f are distinct is called a hexagon triple. The 3-cycle (a, c, e) is called an inside 3-cycle; and the 3-cycles (a, b, c), (c, d, e), and (e, f, a) are called outside 3-cycles. A hexagon triple system of order v is a pair $(X, \mathscr{C})$, where $\mathscr{C}$ is a collection of edge disjoint hexagon triples which partitions the edge set of $3K_v$. Note that the outside 3-cycles form a 3-fold triple system. If the hexagon triple system has the additional property that the collection of inside 3-cycles (a, c, e) is a Steiner triple system it is said to be perfect. In 2004, Küçükçifçi and Lindner had shown that there is a perfect hexagon triple system of order v if and only if $v \equiv 1, 3 \pmod 6$ and $v \geq 7$. In this paper, we investigate the existence of a perfect hexagon triple system with a given subsystem. We show that there exists a perfect hexagon triple system of order v with a perfect sub-hexagon triple system of order u if and only if $v \geq 2u + 1$, $v, u \equiv 1, 3 \pmod 6$ and $u \geq 7$, which is a perfect hexagon triple system analogue of the Doyen–Wilson theorem.

1. Introduction

Let v, λ be two positive integers and K be a set of positive integers. A *pairwise balanced design* (v, K, λ)-PBD is a pair $(X, \mathscr{A})$, where X is a v-set and $\mathscr{A}$ is a family of subsets of X (called *blocks*, the size in K) such that every unordered pair of points is contained in exactly λ blocks of $\mathscr{A}$. The integer v is called the order of the PBD. When $K = \{k\}$, we usually refer to a $(v, \{k\}, \lambda)$-PBD as a *balanced incomplete block design*, denoted by (v, k, λ)-BIBD. A λ-fold triple system of order v (denoted by TS(v, λ)) is just a $(v, 3, \lambda)$-BIBD. When $\lambda = 1$ we have a Steiner triple system (STS(v)) (see, [4]).

The graph consisting of the three 3-cycles (or triples) (a, b, c), (c, d, e), and (e, f, a), where a, b, c, d, e and f are distinct is called a *hexagon triple* (shown in Fig. 1), and denoted by any cyclic 2-shift of $[a, b, c, d, e, f]$ or $[a, f, e, d, c, b]$. The 3-cycle (a, c, e) is called an *inside* 3-cycle; and the 3-cycles (a, b, c), (c, d, e), and (e, f, a) are called *outside* 3-cycles. A *hexagon triple system* of order v is a pair $(X, \mathscr{C})$, where $\mathscr{C}$ is a collection of edge disjoint hexagon triples which partitions the edge set of $3K_v$ with vertex set X, where $3K_v$ is the complete graph with v vertices and index 3. Note that the outside 3-cycles form a 3-fold triple system (TS$(v, 3)$). A hexagon triple system is said to be *perfect* if the collection of inside 3-cycles is a Steiner triple system (STS(v)). In what follows, we will use the notation PHTS(v) to denote a perfect hexagon triple system of order v. In 2004, Küçükçifçi and Lindner [9] completely determined the spectrum for perfect hexagon triple systems.

Theorem 1.1 (*[9, Küçükçifçi and Lindner]*). *There exists a* PHTS(v) *if and only if* $v \equiv 1, 3 \pmod 6$, *and* $v \geq 7$.

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be a PHTS(v) and a PHTS(u), respectively. If $Y \subset X$ and $\mathscr{B} \subset \mathscr{A}$, then $(Y, \mathscr{B})$ is called a *subsystem* of $(X, \mathscr{A})$, or $(Y, \mathscr{B})$ is said to be *embedded* in $(X, \mathscr{A})$. Note that a hexagon triple consists of three triples, so we can think of a hexagon triple system as the piecing together of the triples of a 3-fold triple system into hexagon triples. Hence, if there is a

* Corresponding author.
E-mail addresses: gh19860109@163.com (H. Gao), jhwang@ntu.edu.cn (J. Wang).

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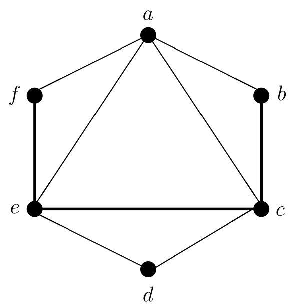

Fig. 1. Hexagon triple.

PHTS(v) containing a PHTS(u) as a subsystem, then there are a TS(v, 3) containing a TS(u, 3) as a subsystem and an STS(v) containing an STS(u) as a subsystem.

The problem of constructing (v, k, λ)-BIBD containing subsystems has been extensively studied by many researchers (see, for example [1,6,8,11,12,14,15] and references therein). In 1973, Doyen and Wilson [6] proved that the necessary conditions for the existence of an STS(v) containing an STS(u) as a subsystem were also sufficient. The result is known as the Doyen–Wilson theorem.

Theorem 1.2 (*[6, Doyen and Wilson]*). *There exists an* STS(v) *containing an* STS(u) *as a subsystem if and only if* $v \geq 2u + 1$ *and* $v, u \equiv 1, 3 \pmod 6$.

In 1980, Stern and Lenz [14] generalized the Doyen and Wilson theorem to all indices and gave a different proof of this result using graph-theoretic methods. In 1989, Stinson [15] gave another proof for the Doyen–Wilson theorem, which was a recursive and completely design-theoretic method by utilizing designs with "holes". So far, the spectra of various triple systems (for example, Kirkman triple systems, nearly Kirkman triple systems, resolvable triple systems, nested Steiner triple systems, perfect dexagon triple systems, etc.) with subsystems have been established by many researchers. For details, we refer the readers to [5,10,13,16–18].

In this paper, we are interested in determining the spectrum of PHTS(v)s with a given subsystem. Since a PHTS(v) containing a PHTS(u) as a subsystem has a TS(v, 3) containing a TS(u, 3) as a subsystem and an STS(v) containing an STS(u) as a subsystem. So we have the following necessary conditions for the existence of a PHTS(v) containing a PHTS(u) as a subsystem.

Lemma 1.3. *The necessary conditions for the existence of a* PHTS(v) *containing a* PHTS(u) *as a subsystem are* $v \geq 2u + 1$, $v, u \equiv 1, 3 \pmod 6$, *and* $u \geq 7$.

The purpose of the present paper is to study the embedding problem of perfect hexagon triple systems, and to establish the perfect hexagon triple system analogue of the Doyen–Wilson theorem.

2. Designs with holes and basic constructions

In this section, we first give the definitions of various types of designs with "holes", and then present some basic constructions for perfect hexagon triple systems with subsystems.

We have already defined perfect hexagon triple systems with subsystems. If we allow the subsystem to be missing (i.e. a hole), we have an incomplete PHTS, or IPHTS. In this paper, we will make extensive use of perfect hexagon triple systems with holes of various types.

An *incomplete perfect hexagon triple system* (IPHTS) is a triple $(X, Y, \mathscr{D})$, where X is a set of points, Y is a subset of X, and $\mathscr{D}$ is a collection of edge disjoint hexagon triples such that every pair of points $\{x, y\}$ as an edge occurs together both in exactly three hexagon triples, and in a unique inside 3-cycle of the hexagon triples, unless $\{x, y\} \subseteq Y$, in which case the pair as an edge occurs in no hexagon triples. Hence, Y is the hole. We say that $(X, Y, \mathscr{D})$ is a (v, u)-IPHTS if $|X| = v$, $|Y| = u$. It is obvious that we can fill in the hole of an IPHTS, as follows.

Construction 2.1. Suppose $(X, Y, \mathscr{D})$ is an IPHTS and $(Y, \mathscr{B})$ is a PHTS. Then $(X, \mathscr{D} \bigcup \mathscr{B})$ is a PHTS.

Construction 2.2. Suppose $(X, Y_1, \mathscr{D})$ and $(Y_1, Y_2, \mathscr{B})$ are IPHTSs. Then $(X, Y_2, \mathscr{D} \bigcup \mathscr{B})$ is an IPHTS.

To prove our main theorem, we also need the concept of perfect group divisible hexagon triple systems which plays an important role in the construction of PHTSs with subsystems. To do this, we first review the concept of a (incomplete) group divisible design.

A *group divisible design* (GDD) with index λ is a triple $(X, \mathscr{G}, \mathscr{B})$, where X is a set of points, $\mathscr{G}$ is a partition of X into subsets (called *groups*), and $\mathscr{B}$ is a collection of subsets (called *blocks*) of X such that every pair of points from distinct groups occurs in exactly λ blocks, and every pair of points from the same group occurs in no blocks.