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The metric dimension of the lexicographic product of graphs

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ABSTRACT

For a set W of vertices and a vertex v in a connected graph G, the k-vector $r_W(v) = (d(v, w_1), \ldots, d(v, w_k))$ is the *metric representation* of v with respect to W, where $W = \{w_1, \ldots, w_k\}$ and $d(x, y)$ is the distance between the vertices x and y. The set W is a *resolving set* for G if distinct vertices of G have distinct metric representations with respect to W. The minimum cardinality of a resolving set for G is its *metric dimension*. In this paper, we study the metric dimension of the lexicographic product of graphs G and H, denoted by $G[H]$. First, we introduce a new parameter, the *adjacency dimension*, of a graph. Then we obtain the metric dimension of $G[H]$ in terms of the order of G and the adjacency dimension of H.

1. Introduction

In this section, we present some definitions and known results that are necessary to prove our main theorems. Throughout this paper, G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use $\overline{G}$ for the complement of G. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path joining u and v in G. Also, $N_G(v)$ is the set of all neighbors of vertex v in G. We write these simply as $d(u, v)$ and $N(v)$ when no confusion can arise. The adjacency and non-adjacency relations are denoted by $\sim$ and $\not\sim$, respectively. We use P_n and C_n to denote the isomorphism classes of n-vertex paths and cycles, respectively. We use $\langle v_1, \ldots, v_n\rangle$ and $[v_1, \ldots, v_n]$ to denote specific n-vertex paths and cycles with vertices $v_1, \ldots, v_n$ in order. We also use notation $\mathbf{1}$ for the vector $(1, \ldots, 1)$ and $\mathbf{2}$ for the vector $(2, \ldots, 2)$.

For $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r_W(v) = (d(v, w_1), \ldots, d(v, w_k))$$

is the *metric representation* of v with respect to W. The set W is a *resolving set* for G if the vertices of G have distinct metric representations. In this case, we say that W *resolves* G. Elements in a resolving set are *landmarks*. A resolving set W for G with minimum cardinality is a *metric basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\mu(G)$. The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [15] and by Harary and Melter [11]. For more results related to these concepts see [2,3,7,9,17].

We say that a set W *resolves* a set T of vertices in G if the metric representations of vertices in T with respect to W are distinct. When $W = \{x\}$, we say that the vertex x resolves T. To determine whether a given set W is a resolving set for G, it is sufficient to look at the metric representations of vertices in $V(G) \setminus W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$.

Two distinct vertices u and v are *twins* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. We write $u \equiv v$ if and only if $u = v$ or u and v are twins. In [12], it is proved that "$\equiv$" is an equivalence relation. The equivalence class of vertex v is denoted by v^*. Hernando

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et al. [12] proved that v^* is a clique or an independent set in G. As in [12], we say that v^* is of type 1, K, or N if v^* is a class of size 1, a clique of size at least 2, or an independent set of size at least 2, respectively. We denote the number of equivalence classes of G with respect to "$\equiv$" by $\iota(G)$. We denote by $\iota_K(G)$ and $\iota_N(G)$ the number of classes of type K and type N in G, respectively. We also use $a(G)$ and $b(G)$ for the number of vertices in G belonging to classes of type K or type N, respectively. Clearly, $\iota(G) = n(G) - a(G) - b(G) + \iota_N(G) + \iota_K(G)$.

Observation 1.1 *([12]). If u and v are twins in a graph G, and W resolves G, then u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves G.*

Theorem 1.2 *([8]). If G is a connected graph of order n, then*

(i) $\mu(G) = 1$ *if and only if* $G = P_n$ *and*
(ii) $\mu(G) = n - 1$ *if and only if* $G = K_n$.

Let G and H be two graphs with disjoint vertex sets. The *join* of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv\colon u \in V(G), v \in V(H)\}$.

Theorem 1.3 *([4,5]).*

(i) *If* $n \notin \{3, 6\}$, *then* $\mu(C_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$,
(ii) *If* $n \notin \{1, 2, 3, 6\}$, *then* $\mu(P_n \vee K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

The *Cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H) = \{(v, u)\colon v \in V(G), u \in V(H)\}$, where two vertices (v, u) and (v', u') are adjacent if $u = u'$ and $vv' \in E(G)$, or $v = v'$ and $uu' \in E(H)$. The metric dimension of the Cartesian product of graphs is studied by Caceres et al. in [6]. They obtained the metric dimension of $G \square H$ when $G, H \in \{P_n, C_n, K_n\}$.

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (v, u) and (v', u') are adjacent if $vv' \in E(G)$, or $v = v'$ and $uu' \in E(H)$. When the order of G is at least 2, it is easy to see that $G[H]$ is connected if and only if G is connected. For more information about the lexicographic product of graphs, see [13].

This paper studies the metric dimension of the lexicographic product of graphs. The main goal of Section 2 is introducing a new parameter called the "adjacency dimension". In Section 3, we determine the metric dimension of some lexicographic products of the form $G[H]$ in terms of the order of G and the adjacency dimension of H. In Corollaries 3.12 and 3.13, we use Theorems 3.3, 3.5, 3.7 and 3.9 to obtain the exact value of the metric dimension of $G[H]$, where $G = C_n$ for $n \geq 5$ or $G = P_n$ for $n \geq 4$, and $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_{m_1,\ldots,m_t}, \overline{K}_{m_1,\ldots,m_t}\}$.

2. Adjacency resolving sets

Khuller et al. [14] considered the application of the metric dimension of a connected graph in robot navigation. In that sense, a robot moves from node to node of a graph. If the robot knows its distances to a sufficiently large set of landmarks, then its position on the graph is uniquely determined. This suggests the problem of finding the minimum number of landmarks needed, and where they should be located, so that the distances to the landmarks uniquely determine the robot's position on the graph. The solutions of these problems are the *metric dimension* and a *metric basis* of the graph, respectively.

Now let there exist a large number of landmarks, but suppose that the cost of computing distance is too much for the robot. In this case, we want the robot to be able to determine its position only from landmarks adjacent to it. Now the problem is that of finding the minimum number of landmarks needed, and where they should be located, so that adjacency and non-adjacency to the landmarks uniquely determine the robot's position on the graph. This problem motivates introducing *adjacency resolving sets* in graphs.

Definition 2.1. Let G be a graph, and let $W = \{w_1, \ldots, w_k\} \subseteq V(G)$. For each vertex $v \in V(G)$, the *adjacency representation* of v with respect to W is the k-vector

$$\hat{r}_W(v) = (a_G(v, w_1), \ldots, a_G(v, w_k)),$$

where

$$a_G(v, w_i) = \begin{cases} 0 & \text{if } v = w_i, \\ 1 & \text{if } v \sim w_i, \\ 2 & \text{if } v \nsim w_i. \end{cases}$$

The set W is an *adjacency resolving set* for G if the vectors $\hat{r}_W(v)$ for $v \in V(G)$ are distinct. The minimum cardinality of an adjacency resolving set is the *adjacency dimension* of G, denoted by $\hat{\mu}(G)$. An adjacency resolving set of cardinality $\hat{\mu}(G)$ is an *adjacency basis* of G.