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A note on the Entropy/Influence conjecture

Nathan Keller [a], Elchanan Mossel [b,a,*], Tomer Schlank [c]

[a] Weizmann Institute of Science, Israel
[b] U.C. Berkeley, United States
[c] Hebrew University, Israel

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ABSTRACT

The Entropy/Influence conjecture, raised by Friedgut and Kalai (1996) [9], seeks to relate two different measures of concentration of the Fourier coefficients of a Boolean function. Roughly saying, it claims that if the Fourier spectrum is "smeared out", then the Fourier coefficients are concentrated on "high" levels. In this note we generalize the conjecture to biased product measures on the discrete cube.

1. Introduction

Definition 1.1. Consider the discrete cube $\{0, 1\}^n$ endowed with the product measure $\mu_p = (p\delta_{\{1\}} + (1-p)\delta_{\{0\}})^{\otimes n}$, denoted in the sequel by $\{0, 1\}^n_p$, and let $f : \{0, 1\}^n_p \to \mathbb{R}$. The Fourier–Walsh expansion of f with respect to the measure μ_p is the unique expansion

$$f = \sum_{S \subset \{1,2,\ldots,n\}} \alpha_S u_S,$$

where for any $T \subset \{1, 2, \ldots, n\}$,[1]

$$u_S(T) = \left(-\sqrt{\frac{1-p}{p}}\right)^{|S\cap T|} \left(\sqrt{\frac{p}{1-p}}\right)^{|S\setminus T|}.$$

In particular, for the uniform measure (i.e., $p = 1/2$), $u_S(T) = (-1)^{|S\cap T|}$. The coefficients α_S are denoted by $\hat{f}(S)$,[2] and the level of the coefficient $\hat{f}(S)$ is $|S|$.

Properties of the Fourier–Walsh expansion are one of the main objects of study in discrete harmonic analysis. The Entropy/Influence (EI) conjecture, raised by Friedgut and Kalai [9] in 1996, seeks to relate two measures of concentration of

* Corresponding author at: U.C. Berkeley, United States.
E-mail addresses: nathan.keller@weizmann.ac.il (N. Keller), mossel@stat.berkeley.edu (E. Mossel), tomer.schlank@gmail.com (T. Schlank).

1 Throughout the paper, we identify elements of $\{0, 1\}^n$ with subsets of $\{1, 2, \ldots, n\}$ in the natural way.

2 Note that since the functions $\{u_S\}_{S\subset\{1,\ldots,n\}}$ form an orthonormal basis, the representation is indeed unique, and the coefficients are given by the formula $\hat{f}(S) = \mathbb{E}_{\mu_p}[f \cdot u_S]$.

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the Fourier coefficients (i.e. coefficients of the Fourier–Walsh expansion) of Boolean functions. The first of them is *spectral entropy*.

Definition 1.2. Let $f : \{0,1\}_p^n \to \{-1,1\}$ be a Boolean function. The spectral entropy of f with respect to the measure μ_p is

$$\mathrm{Ent}_p(f) = \sum_{S \subset \{1,\dots,n\}} \hat{f}(S)^2 \log\left(\frac{1}{\hat{f}(S)^2}\right),$$

where the Fourier–Walsh coefficients are computed w.r.t. to μ_p.

Note that by Parseval's identity, for any Boolean function we have $\sum_S \hat{f}(S)^2 = 1$, and thus, the squares of the Fourier coefficients can be viewed as a probability distribution on the set $\{0,1\}^n$. In this notation, the spectral entropy is simply the entropy of this distribution, and intuitively, it measures how much the Fourier coefficients are "smeared out".

The second notion is *total influence*.

Definition 1.3. Let $f : \{0,1\}_p^n \to \{0,1\}$. For $1 \le i \le n$, the influence of the i-th coordinate on f with respect to μ_p is

$$\mathrm{I}_i^p(f) = \Pr_{x \sim \mu_p}[f(x) \neq f(x \oplus e_i)],$$

where $x \oplus e_i$ denotes the point obtained from x by replacing x_i with $1 - x_i$ and leaving the other coordinates unchanged. The total influence of the function f is

$$\mathrm{I}_p(f) = \sum_{i=1}^{n} \mathrm{I}_i^p(f).$$

Influences of variables on Boolean functions were studied extensively in the last decades, and have applications in a wide variety of fields, including Theoretical Computer Science, Combinatorics, Mathematical Physics, Social Choice Theory, etc. (See, e.g., the survey [12].) As observed in [10], the total influence can be expressed in terms of the Fourier coefficients:

Observation 1.4. *Let* $f : \{0,1\}_p^n \to \{-1,1\}$*. Then*

$$\mathrm{I}_p(f) = \frac{1}{4p(1-p)} \sum_S |S| \hat{f}(S)^2. \tag{1}$$

In particular, for the uniform measure $\mu_{1/2}$, $\mathrm{I}_{1/2}(f) = \sum_S |S|\hat{f}(S)^2$.

Thus, in terms of the distribution induced by the Fourier coefficients, the total influence is (up to normalization) the *expectation of the level* of the coefficients, and it measures the question whether the coefficients are concentrated on "high" levels.

The Entropy/Influence conjecture asserts the following:

Conjecture 1.5 (*Friedgut and Kalai*)**.** *Consider the discrete cube* $\{0,1\}^n$ *endowed with the uniform measure* $\mu_{1/2}$*. There exists a universal constant c, such that for any n and for any Boolean function* $f : \{0,1\}_{1/2}^n \to \{-1,1\}$,

$$\mathrm{Ent}_{1/2}(f) \le c \cdot \mathrm{I}_{1/2}(f).$$

The conjecture, if confirmed, has numerous significant implications. For example, it would imply that for any property of graphs on n vertices, the sum of influences is at least $c(\log n)^2$ (which is tight for the property of containing a clique of size $\approx \log n$). The best currently known lower bound, by Bourgain and Kalai [5], is $\Omega((\log n)^{2-\epsilon})$, for any $\epsilon > 0$.

Another consequence of the conjecture would be an affirmative answer to a variant of a conjecture of Mansour [14] stating that if a Boolean function can be represented by a DNF formula of polynomial size in n (the number of coordinates), then most of its Fourier weight is concentrated on a polynomial number of coefficients (see [15] for a detailed explanation of this application). This conjecture, raised in 1995, is still wide open.

The main object of this note is to generalize the Entropy/Influence conjecture to the product measure μ_p on the discrete cube. We state a generalization of the conjecture to the biased case:

Conjecture 1.6. *There exists a universal constant c, such that for any* $0 < p < 1$*, for any n and for any Boolean function* $f : \{0,1\}_p^n \to \{-1,1\}$,

$$\mathrm{Ent}_p(f) \le cp\log(1/p) \cdot \mathrm{I}_p(f).$$