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Intrinsically linked signed graphs in projective space[☆]

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ABSTRACT

We define a signed embedding of a signed graph into real projective space to be an embedding such that an embedded cycle is 0-homologous if and only if it is balanced. We characterize signed graphs that have a linkless signed embedding. In particular, we exhibit 46 graphs that form the complete minor-minimal set of signed graphs that contain a non-split link for every signed embedding. With one trivial exception, these graphs are derived from different signings of the seven Petersen family graphs.
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1. Introduction

Recall that a graph is *intrinsically linked* if every embedding of the graph in $\mathbb{R}^3$ contains at least two non-splittably linked cycles. The set of all minor-minimal intrinsically linked graphs is given by the seven Petersen family graphs [3,11,12]. These graphs are obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges. We denote these by K_6, P_7, P_8, P_9, P_{10}, $K_{4,4} \setminus e$, and $K_{3,3,1}$, where P_{10} is the classic Petersen graph (see Appendix A).

We say that a graph is *intrinsically linked in* $\mathbb{R}P^3$ provided every embedding into $\mathbb{R}P^3$ contains at least one pair of non-splittably linked cycles (see the next section for a formal definition of non-splittably linked cycles in $\mathbb{R}P^3$). Note that every spatially embedded graph corresponds to a projectively embedded graph. We also note that some intrinsically linked spatial graphs have linkless embeddings in $\mathbb{R}P^3$ (for example, K_6, see [2]). As a result of Robertson and Seymour's Minor Theorem [10], the collection of minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$ is finite. Bustamante et al., [2] fully characterized intrinsically linked graphs in $\mathbb{R}P^3$ with connectivity 0, 1, and 2 and in all, 597 graphs were shown to be minor-minimal intrinsically linked in $\mathbb{R}P^3$, although a complete classification is still unknown.

A *signed embedding* of a signed graph $\Sigma = (\Gamma, \sigma)$ is an embedding into $\mathbb{R}P^3$ for which a cycle is 1-homologous if and only if its sign is negative. An *intrinsically linked signed graph* denotes a signed graph for which every signed embedding contains at least one pair of non-splittably linked cycles. In this paper, we seek to classify all minor-minimal intrinsically linked signed graphs in $\mathbb{R}P^3$. This is motivated by Zaslavsky's approach for projective planar graphs [15]. Zaslavsky found

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eight signed forbidden minors for projective planarity, significantly less than the set of 35 unsigned forbidden minors for projective planarity found by Archdeacon [1] and Glover et al. [5]. We seek the analogous set of graphs for intrinsically linked graphs in projective space, $\mathbb{R}P^3$, and have found a complete set of 46 such graphs. These 46 graphs are V_2^o, the seven balanced Petersen family graphs, 32 Petersen family graphs with a balancing vertex, and six Petersen family graphs that have all positive edges except for one 3-cycle with all edges negative.

1.1. Definitions and notation

Throughout, let M be a 3-manifold. We define a *graph* G as a set of vertices $V(G)$ and edges $E(G)$, where an *edge* is an unordered pair (v_i, v_j) with $v_i, v_j \in V$. It is not necessary that $i \neq j$. Our graphs are finite and are allowed to have loops and redundant edges. Let $v_1, v_2, \ldots, v_n \in V(G)$, with $n \geq 3$, and let

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in E(G)$$

such that $v_i \neq v_j$ for $i \neq j$, then the sequence of vertices $v_1, v_2, \ldots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ is an *n-cycle* in the graph G. A *loop* at vertex v is an edge of the form (v, v).

A *link* is two or more disjointly embedded circles (cycles) in M. Let $L_i \cup L_j$ be a 2-component link, where $i \neq j$. We say $L_i \cup L_j$ is *splittable* if there exists $A \subseteq M$, where A is homeomorphic to B^3, such that $L_i \subseteq A$ and $L_j \subseteq A^C$. Otherwise, $L_i \cup L_j$ is a *non-splittable* link. If every embedding of G into M contains a pair of cycles that form a non-splittable two-component link, then G is *intrinsically linked* in M. In this paper, we will use "intrinsically linked" to mean intrinsically linked in $\mathbb{R}P^3$. Similarly, when we say "embedded" (respectively, "embedding") we mean "embedded (respectively, embedding) in $\mathbb{R}P^3$".

If H is a graph such that H can be obtained from G by a sequence of edge removals, vertex removals, and edge contractions, then H is a *minor* of G, written $H \leq G$. If $H \leq G$ but $H \neq G$, then H is a *proper minor* of G, written $H < G$. If $e \in E(G)$ is the edge contracted or deleted in G to obtain H, we write that $H = G/e$ or $H = G \setminus e$, respectively. To contract an edge $e = (v, w)$, replace e with the new vertex v_e, which becomes adjacent to all of the former neighbors of v and w. Note that if H, a minor of a graph G, is intrinsically linked, then G is as well [9].

A *signed graph* $\Sigma = (G, \sigma)$ consists of a graph G and an edge signing $\sigma : E(G) \to \{+, -\}$. We also denote G as $|\Sigma|$. The sign of a cycle $C = (v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) = e_1e_2 \cdots e_{n-1}e_n$ is obtained by multiplying the signs of its constituent edges:

$$\sigma(C) = \sigma(e_1)\sigma(e_2) \cdots \sigma(e_n).$$

The sign of a path is defined similarly. A signed graph Σ is *balanced* if all of its cycles are positive. Otherwise, it is *unbalanced*.

Recall that *switching* a signed graph Σ means reversing the signs of all edges between a vertex set $X \subseteq V(\Sigma)$ and its complement. We say Σ_1 and Σ_2 are *switching isomorphic* if one can be obtained from the other by a sequence of switchings. Recall that switching preserves signs of cycles, and two signed graphs are switching isomorphic if and only if all signs of corresponding cycles are the same. We sometimes abuse notation by using Σ in the remainder of the paper to denote a particular signed graph, as well as the switching class of Σ. It is not difficult to see that every balanced graph Σ is switching isomorphic to an all positive signing of its underlying graph, which we denote Σ^+. (Similarly, an all negative signing of Σ is denoted Σ^-.) A vertex v of a signed graph Σ is a *balancing vertex* of Σ if $\Sigma \setminus v$ is balanced, but Σ is not.

Contracting an edge on a signed graph is only allowed after the graph is switched so that the edge being contracted is positive. This is necessary in order to preserve the sign of each cycle. A *minor of a signed graph* $\Sigma = (G, \sigma)$ is any signed graph that can be obtained from Σ by a sequence of switchings, vertex deletions, edge deletions, and edge contractions. Note that Zaslavsky [15,14] used the term "link minor" to denote our version of "minor". We use the shorter term so as not to overuse the term "link".

Recall that *real projective space*, denoted $\mathbb{R}P^3$, is the sphere S^3 with identified opposite points, or equivalently as the closed 3-ball B^3 with antipodal boundary points identified. All of our embedded graphs and ambient isotopies are in the PL category, which is possible as $\mathbb{R}P^3$ is a quotient of S^3. An embedded cycle is 0-*homologous* (0-H) if it crosses the boundary an even number of times and 1-*homologous* (1-H) if it crosses the boundary an odd number of times. We say that an embedded graph is *affine* if it is contained in a 3-ball in $\mathbb{R}P^3$.

An important signed graph, V_2^o, is the graph consisting of two disjoint loops, each signed negatively.

2. Preliminary results

Lemma 1. *If Σ has a linkless embedding, then so does every minor of Σ.*

Proof. Embed Σ linklessly in projective space. Consider an edge e. If necessary, switch so that e has positive sign (this corresponds, geometrically, to performing ambient isotopy so that in the embedding, e does not touch the boundary). Contract the edge e to obtain a linkless embedding of Σ/e. Similarly, removing an edge or a vertex results in a linkless embedding. □

It follows [10] that we can indeed characterize the set of minor-minimal intrinsically linked signed graphs by a finite set of minor-minimal graphs with this property. We exhibit one such graph now.

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