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Gap vertex-distinguishing edge colorings of graphs

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ABSTRACT

In this paper, we study a new coloring parameter of graphs called the *gap vertex-distinguishing edge coloring*. It consists in an edge-coloring of a graph G which induces a vertex distinguishing labeling of G such that the label of each vertex is given by the difference between the highest and the lowest colors of its adjacent edges. The minimum number of colors required for a gap vertex-distinguishing edge coloring of G is called the *gap chromatic number* of G and is denoted by $\text{gap}(G)$.

We here study the gap chromatic number for a large set of graphs G of order n and prove that $\text{gap}(G) \in \{n-1, n, n+1\}$.

1. Introduction and definitions

All graphs considered in this paper are finite and undirected. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For any undefined terms, we refer the reader to [4].

A vertex labeling of a graph G is said to be *vertex-distinguishing labeling* if distinct vertices are assigned distinct labels. Let k be a non-negative integer. A *k-edge-coloring* of G is a mapping f from $E(G)$ to $\{1, 2, \ldots, k\}$. We say that an edge coloring is *proper* if no two adjacent edges have the same color. Many researchers investigated the question of edge coloring inducing a vertex distinguishing labeling. This is often referred to as *vertex-distinguishing edge colorings*. In the literature, four main different functions have been proposed to label each vertex v of G according to the colors of its incident edges. A vertex labeling l induced by an edge-coloring f is said to be:

(1) vertex-labeling by sum if $l(v) = \sum_{v \ni e} f(e), \forall v \in V$ (see [11,2]).
(2) vertex-labeling by sets if $l(v) = \bigcup_{v \ni e} f(e), \forall v \in V$ (see [8,9,14]).
(3) vertex-labeling by multiset if $l(v) = \biguplus_{v \ni e} f(e), \forall v \in V$ (see [1,6,7,10]).
(4) vertex-labeling by product if $l(v) = \prod_{v \ni e} f(e), \forall v \in V$ (see [17]).

The problem of vertex-distinguishing edge colorings offers many variants and received a great interest during these last years. We refer the interested reader to Chapter 13 of Chartrand and Zhang's book [12]. In this paper, we define a new variant called *gap vertex-distinguishing edge coloring*, which is defined as follows.

Definition 1. Let G be a graph, k be a positive integer and f be a mapping from $E(G)$ to the set $\{1, 2, \ldots, k\}$. For each vertex v of G, the label of v is defined as

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$$l(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1 \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise.} \end{cases}$$

The mapping f is called gap vertex-distinguishing labeling if distinct vertices have distinct labels. Such a coloring is called a *gap-k-coloring*.

The minimum positive integer k for which G admits a gap-k-coloring is called the *gap chromatic number* of G and is denoted by $\text{gap}(G)$. Necessary and sufficient conditions for the existence of such a coloring are given by the following proposition.

Proposition 1. *A graph G admits a gap vertex-distinguishing edge coloring if and only if it has no connected component isomorphic to K_1 or K_2.*

Proof. Since no isolated vertex of a graph G is assigned a label in an edge coloring of G, we may assume that G has no isolated vertices. Furthermore, if G contains a connected component K_2, then the two vertices of K_2 are assigned the same label in any edge coloring of G. Hence, when considering gap vertex-distinguishing edge coloring of a graph G, we may assume that the order of every connected component of G is at least 3. Let G be such a graph and let $E(G) = \{e_1, e_2, \dots, e_m\}$. The following edge coloring function: $f(e_i) = 2^{i-1}$ for $1 \leqslant i \leqslant m$ induces a gap vertex-distinguishing edge coloring of G. □

The following lemma gives a lower bound on the gap chromatic number.

Lemma 2. *A graph G of order n and without connected component isomorphic to K_1 or K_2 satisfies* $\text{gap}(G) \geq n - 1$*. Moreover, if $\delta(G) \geq 2$ or if any vertex of degree greater than 1 has at least two adjacent vertices of degree 1, then* $\text{gap}(G) \geq n$*.*

To illustrate these concepts, consider the graph G shown in Fig. 1(a). A 6-edge coloring f_1 of G is given in Fig. 1(b) and a 5-edge coloring f_2 of G is given in Fig. 1(c). For example, in Fig. 1(b), the vertex w is incident to two edges colored 2 and one edge colored 3, then $l_1(w) = 1$, while the vertex z is incident with one edge colored 6, then $l_1(z) = 6$. The resulting vertex labels are distinct for both figures. By Lemma 2, we have $\text{gap}(G) \geq 5$; hence we can immediately conclude that $\text{gap}(G) = 5$.

After a strong analysis of this problem, we raised the conjecture asserting that there is no graph G of order n with $\text{gap}(G) > n + 1$.

Conjecture 3. *A graph G of order n (not necessarily connected), without isolated edges and isolated vertices has* $\text{gap}(G) \in \{n - 1, n, n + 1\}$.

In the following sections, we prove this conjecture for a large set of graphs and we even decide the exact value of $\text{gap}(G)$. The rest of the paper is organized as follows: first, we point out some previous work related to the topic of this paper and give some motivations to investigate this new parameter. The results of Section 3 will confirm our conjecture for a large part of graphs with minimum degree at least 2. In Section 4, we prove our conjecture for some classes of graphs with minimum degree 1, such as paths, complete binary trees and all trees with at least two leaves at distance 2. This classification of our results according to $\delta(G)$ is due to the definition of our parameter, especially to the definition of labels of vertices of degree one. Concluding remarks and some open issues are discussed in the last section.

2. Motivation and related work

In this section, we describe the motivation to study the gap coloring problem. We first introduce the following notation: given a set S of positive integers, we denote by $\text{diam}(S)$ the diameter of S, where $\text{diam}(S) = \max\{x - y : x, y \in S\}$. The following proposition is thus obvious.

Proposition 4. *Let S_1 and S_2 be two sets of positive integers, if* $\text{diam}(S_1) \neq \text{diam}(S_2)$*, then $S_1 \neq S_2$.*

From the gap vertex labeling function (Definition 1), we observe that the label of every vertex v with degree at least 2 is the diameter of the set of colors incident to v. Note that this is not the case for the vertices of degree 1. Then, the gap labeling of a graph G can be seen as a strong version of set and multiset labelings (defined on page 2, in (2) and (3)). Indeed, according to Proposition 4, a gap distinguishing labeling of a graph G is also a multiset distinguishing labeling of G and a set distinguishing labeling (if $\delta(G) > 1$). We here present the main results about these related coloring problems.

Let $\chi'_s(G)$ denote the minimum number of colors required to have a proper edge coloring of G that induces a vertex-distinguishing labeling by sets. This coloring number was introduced and studied by Burris and Schelp in [5,8], and independently called *observability* of a graph by Cerny et al. [9]. The following result has been conjectured by Burris and Schelp [8] and proved in [3].

Theorem 5 *([3]). A graph G with n vertices, without isolated edges and with at most one isolated vertex, has $\chi'_s(G) \leq n + 1$.*