Discrete Mathematics 310 (2010) 2499–2503

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Vertex-disjoint triangles in $K_{1,t}$-free graphs with minimum degree at least t

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ARTICLE INFO

Article history:
Received 31 December 2009
Received in revised form 5 June 2010
Accepted 7 June 2010
Available online 5 July 2010

Keywords:
$K_{1,t}$-free graph
Disjoint
Triangle
Ramsey number

ABSTRACT

A graph is said to be $K_{1,t}$-free if it does not contain an induced subgraph isomorphic to $K_{1,t}$. Let $h(t, k)$ be the smallest integer m such that every $K_{1,t}$-free graph of order greater than m and with minimum degree at least t contains k vertex-disjoint triangles. In this paper, we obtain a lower bound of $h(t, k)$ by a constructive method. According to the lower bound, we totally disprove the conjecture raised by Hong Wang [H. Wang, Vertex-disjoint triangles in claw-free graphs with minimum degree at least three, Combinatorica 18 (1998) 441–447]. We also obtain an upper bound of $h(t, k)$ which is related to Ramsey numbers $R(3, t)$. In particular, we prove that $h(4, k) = 9(k - 1)$ and $h(5, k) = 14(k - 1)$.

1. Introduction

In this paper, all graphs are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of G. If $uv \in E(G)$, then u is said to be the *neighbor* of v. We use $N(v)$ to denote the set of neighbors of a vertex v. The *degree* $d(v) = |N(v)|$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. The *join* $G = G_1 \vee G_2$ of graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph $G_1 \bigcup G_2$ together with all the edges jointing V_1 and V_2. For any positive integers k and l, the *Ramsey number* $R(k, l)$ is the smallest integer n such that every graph on n vertices contains either a clique of k vertices or an independent set of l vertices. A (k, l)-*Ramsey graph* is a graph on $R(k, l) - 1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices. By the definition of $R(k, l)$, (k, l)-Ramsey graph does exist for all $k \geq 2$ and $l \geq 2$. The graph C_k is a cycle with k vertices and we call C_3 a triangle. We use mQ to represent m vertex-disjoint copies of graph Q. Other notations can be found in [1].

$K_{1,t}$ is the star of order $t + 1$. A graph is said to be $K_{1,t}$-free if it does not contain an induced subgraph isomorphic to $K_{1,t}$ $(t \geq 2)$. Let $h(t, k)$ be the smallest integer m such that every $K_{1,t}$-free graph of order greater than m and with minimum degree at least t contains k vertex-disjoint triangles. Wang [5] proved that $h(3, k) = 6(k - 1)$ for any $k \geq 2$, and he put forward the following conjecture.

Conjecture 1 ([5]). *For each integer $t \geq 4$, there exists an integer k_t depending on t only such that $h(t, k) = 2t(k - 1)$ for all integers $k \geq k_t$.*

In Section 2, we get a proper lower bound of $h(t, k)$ by a constructive method that $h(4, k) \geq 9(k - 1)$ and $h(t, k) \geq (4t - 9)(k - 1)$ for any $t \geq 5$. Since $4t - 9 > 2t$ for any $t \geq 5$, we totally disprove Conjecture 1. In Section 3, we give an upper bound of $h(t, k)$, which is related to $R(3, t)$. In particular, we prove that $h(4, k) = 9(k - 1)$ and $h(5, k) = 14(k - 1)$. In Section 4, we give some remarks on $h(t, k)$ and list some interesting open problems. The paper ends with one conjecture.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.06.013

2. A lower bound of $h(t, k)$

Let $G_{n,m}$ be the graph whose vertices are $0, 1, \ldots, n-1$ where two vertices i and j are adjacent if and only if $(i-j) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$.

Lemma 1 ([4]). *If $n \geq 6m-2$ then $G_{n,m}$ is a triangle-free regular graph whose degree is equal to $2m$. Furthermore, if $n \leq 8m-3$, then the independent number of $G_{n,m}$ is equal to $2m$.*

Similarly, we define $H_{n,m}$ to be the graph whose vertices are $0, 1, \ldots, n-1$ where two vertices i and j are adjacent if and only if $(i-j) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1), \pm\lfloor \frac{n}{2} \rfloor\}$.

Lemma 2. *$H_{8m-2,m}$ is a triangle-free regular graph whose degree is equal to $2m+1$ and its independent number is equal to $2m+1$.*

Proof. Suppose, to the contrary, that $H_{8m-2,m}$ contains a triangle, say $t_0t_1t_2t_0$ where $0 \leq t_0 < t_1 < t_2 \leq 8m-3$. Then $t_j - t_i \in \{m, m+1, \ldots, 2m-1, 4m-1\}$ for $0 \leq i < j \leq 2$. So $t_2 - t_0 = (t_2 - t_1) + (t_1 - t_0) \geq m + m = 2m$ which implies that $t_2 - t_0 = 4m-1$. Since $t_i \neq t_j$ for $0 \leq i < j \leq 2$, $t_1 - t_0 \leq 2m-1$ and $t_2 - t_1 \leq 2m-1$. This implies that $t_2 - t_0 = (t_2 - t_1) + (t_1 - t_0) \leq 4m-2$, a contradiction. So $H_{8m-2,m}$ is a triangle-free graph.

Let $S = \{\pm m, \pm(m+1), \ldots, \pm(2m-1), 4m-1\}$. Then for any $i, j \in S$, $(i-j) \notin S$. Since $|S| = 2m+1$, $\alpha(H_{8m-2,m}) \geq 2m+1$.

Consider $2m+1$ numbers $0 \leq t_0 < t_1 < \cdots < t_{2m} \leq n-1$ and suppose that $(t_j - t_i) \notin \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$ for any i and j. Put $s_i = t_{i+1} - t_i$ $(i = 1, 2, \ldots, 2m-1)$, $s_0 = n + t_0 - t_{2m}$. It is clear that $s_i \leq m-1$ or $s_i \geq 2m$ for any $i = 0, 1, \ldots, 2m-1$. Let r be equal to the number of members s_i which satisfy $s_i \geq 2m$. If $r \geq 3$, then $n \geq r \cdot 2m + (2m+1-r) \cdot 1 = r(2m-1) + 2m + 1 \geq 8m-2$, that contradicts the assumption of the lemma. If $r \leq 2$ then there exists i such that $s_{i+j} < m$ for every $j = 0, 1, \ldots, m-1$ (we mean that $s_{2m+1} = s_0, s_{2m+2} = s_1, \ldots$). Denote $p_0 = 0$, $p_j = s_i + s_{i+1} + \cdots + s_{i+j-1}$ $(j = 1, 2, \ldots, m)$. Hence $p_j \equiv (t_{i+j} - t_i) (\mathrm{mod}\ n)$. Since every $s_{i+j} \geq 1$, $p_m \geq m$. Let $j = \min\{l : p_l \geq m\}$. So $p_j \geq m$, $p_{j-1} \leq m-1$, $p_j = p_{j-1} + s_{i+j} \leq (m-1) + (m-1) \leq 2m-1$. Therefore, $(t_i - t_{i+j}) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$, which leads to a contradiction. $\square$

Theorem 3. *For each integer $k \geq 2$, $h(4, k) \geq 9(k-1)$.*

Proof. Let W be a wheel of order 9. Label W's center by v_0 and its neighbors by $v_1, v_2, \ldots, v_8$. Let H be a graph obtained from W by adding two edges v_1v_5 and v_2v_6. It is obvious that H does not contain two vertex-disjoint triangles. Set $P(H) = \{v_3, v_4, v_7, v_8\}$. Let Π_k be the set of graphs of order $9(k-1)$ such that a graph G belongs to Π_k if and only if it is obtained from $k-1$ vertex-disjoint copies $H_1, \ldots, H_{k-1}$ of H by adding $2(k-1)$ new edges on $\bigcup_{i=1}^{k-1} P(H_i)$ so that these new edges form a perfect matching. It is easy to check that every graph H belonging to $\prod_k$ is the $K_{1,4}$-free graph which contains at most $k-1$ vertex-disjoint triangles and $\delta(G) \geq 4$. So $h(4, k) \geq 9(k-1)$. $\square$

Theorem 4. *For each integers $t \geq 5$ and $k \geq 2$,*

$$h(t, k) \geq \begin{cases} (4t-6)(k-1), & \text{if } t \text{ is odd;} \\ (4t-9)(k-1), & \text{if } t \text{ is even.} \end{cases}$$

Proof. Let $G = (k-1)(K_1 \vee G_{8m-3,m})$. Then $|V(G)| = (8m-2)(k-1)$ and $\delta(G) = 2m+1$. By Lemma 1, G is a $K_{1,2m+1}$-free graph which contains at most $k-1$ vertex-disjoint triangles. So $h(2m+1, k) \geq (8m-2)(k-1)$. Let $t = 2m+1$. Then $h(t, k) \geq (4t-6)(k-1)$. Similarly, we put $H = (k-1)(K_1 \vee H_{8m-2,m})$. Then $|V(G)| = (8m-1)(k-1)$ and $\delta(G) = 2m+2$. By Lemma 2, H is a $K_{1,2m+2}$-free graph which contains at most $k-1$ vertex-disjoint triangles. So we also have $h(2m+2, k) \geq (8m-1)(k-1)$. Let $t = 2m+2$. Then $h(t, k) \geq (4t-9)(k-1)$. $\square$

By Theorems 3 and 4, we totally disprove Conjecture 1.

3. An upper bound of $h(t, k)$

In this section, we continue to consider $K_{1,t}$-free graphs and give an upper bound of $h(t, k)$. First, we introduce a useful lemma, which is known as *Ramsey's Theorem*.

Lemma 5 ([1] (Ramsey's Theorem)). *For any two integers $k \geq 2$ and $l \geq 2$, $R(k, l) \leq R(k, l-1) + R(k-1, l)$. Furthermore, if $R(k, l-1)$ and $R(k-1, l)$ are both even, then the strict inequality holds.*

In [2] (also see page 7 in [3]), Burr et al. proved that $R(k, t) \geq R(k-1, t) + 2t - 3$ for $k, t \geq 3$. It follows that $R(3, t) \geq R(2, t-1) + 2t - 3 = 3t - 3$ for $t \geq 3$. So we have the following lemma.

Lemma 6. *For each integer $t \geq 4$, $\max\left\{\left\lfloor \frac{3(t-1)}{2} \right\rfloor, 2t-2, \frac{5}{2}t-4, 3t-6\right\} \leq R(3, t-1) + t - 4$.*