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The 2-extendability of 5-connected graphs on the Klein bottle

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ABSTRACT

A graph is said to be *k-extendable* if any independent set of k edges extends to a perfect matching. In this paper, we shall characterize the forbidden structures for 5-connected graphs on the Klein bottle to be 2-extendable. This fact also gives us a sharp lower bound of representativity of 5-connected graphs embedded on the Klein bottle to have such a property, which was considered in Kawarabayashi et al. (submitted for publication) [4].

0. Introduction

A graph in this paper is a simple graph, that is, one with no loops and no multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The number of vertices of G is often called the *order* of G.

A set M of edges in a graph G is said to be a *matching* (or o members of M share a vertex. A matching M is *perfect* if every vertex of G is covered by an edge of M. A graph G with $|V(G)| \geq 2k+2$ is said to be *k-extendable* if every matching $M \subseteq E(G)$ with $|M| = k$, extends to a perfect matching in G.

Plummer [8,9,7] has introduced this notion of k-extendability of graphs and discussed it, combining topological properties. For example in [9], he has proved that every 5-connected planar graph of even order is 2-extendable.

Let G be a graph and $\{e_1, e_2\}$ an independent pair of edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$. If $G - \{u_1, v_1, u_2, v_2\}$ has an *odd component*, that is, a connected component consisting of an odd number of vertices, then the subgraph in G induced by the odd component and $\{u_1, v_1, u_2, v_2\}$ is called a *generalized butterfly*. It is clear that if G contains a generalized butterfly, then G is not 2-extendable; any matching containing the two edges e_1 and e_2 cannot cover all vertices in the odd component. By these facts, Plummer proved the following theorem:

Theorem 1 ([9]). *Every 4-connected maximal planar graph of even order is 2-extendable unless it contains a generalized butterfly.*

More generally, Plummer [7] has shown that for a given closed surface, there exists an upper bound for a natural number k such that the surface admits embeddings of k-extendable graphs and Dean [3] has determined the precise value of the bound. Recently, Aldred et al. [1] have proven that a triangulation of even order on a closed surface of positive genus is 2-extendable if it has sufficiently large representativity. (The representativity of G on a closed surface F^2 denoted by $\gamma(G)$ is defined as follows: $\gamma(G) := \min\{|G \cap \ell| : \ell$ is an essential simple closed curve on $F^2\}$. A graph G on F^2 is said to be *r-representative* if $\gamma(G) \geq r$.) Furthermore in [6], Mizukai et al. have discussed the 2-extendability of 5-connected graphs on the torus

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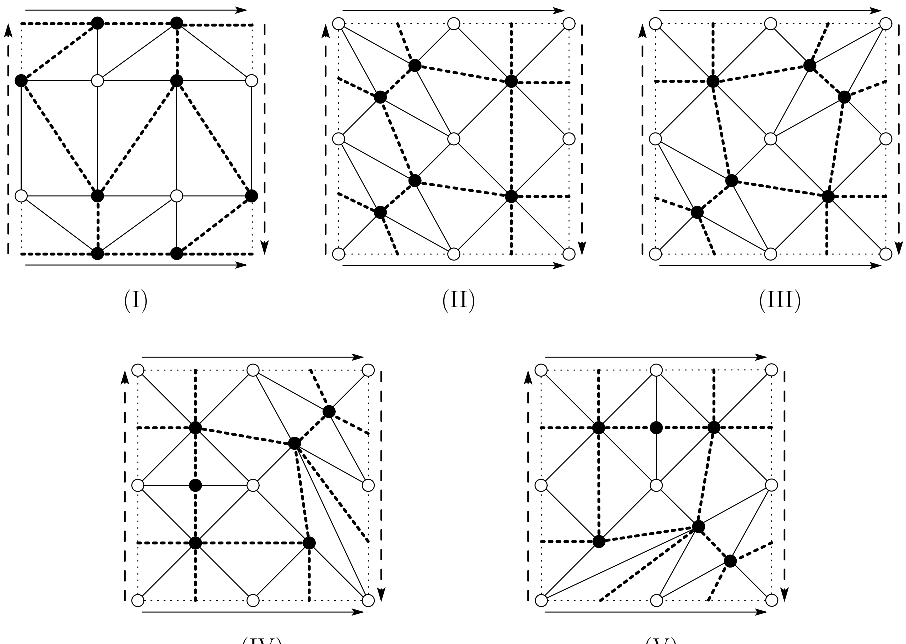

Fig. 1. Forbidden structures for the 2-extendability of 5-connected graphs on K^2.

and characterized the forbidden structures for graphs having such a property. (These topics, which state the extendability of graphs on surfaces with low genera, were also treated in the research of restricted matching extension by Aldred and Plummer [2].) Actually in the proofs in [6], they needed the fact that those graphs in the theorem are 1-extendable; in fact, Thomas and Yu's results in [10,11] guarantee the property.

Although a 5-connected graph on the Klein bottle is not 1-extendable in general, we could prove the following theorem by carrying out topological arguments on the Klein bottle, which is our main result in the paper.

Theorem 2. *A* 5*-connected graph of even order embedded on the Klein bottle is* 2*-extendable if and only if it has none of the structures depicted in Fig. 1.*

In the figure, to obtain the Klein bottle, we have to identify the top and the bottom of the dotted rectangle in parallel, and the right-hand side and left-hand side in anti-parallel. To get an actual graph which are not 2-extendable, replace each of the white vertices with a connected planar graph of odd order and choose additional edges from edges drawn by thick dotted lines so that they include at least one independent pair of edges. The edges between a white vertex and a black vertex may split into several edges with a common black end. Furthermore, the resulting graph should be simple and 5-connected. For example, all dotted edges in each of (I), (II) and (IV) cannot be included simultaneously since if we do so, there would be multiple edges between two black vertices. Therefore, if one wants a triangulation on the Klein bottle, then only (III) and (V) may be used.

By observing those figures, the following corollary related to representativity is an easy consequence:

Corollary 3. *A* 5*-connected and* 4*-representative graph on the Klein bottle with even order is* 2*-extendable.*

1. $\{e_1, e_2\}$-blocker

First we shall prepare two key lemmas to prove Theorem 2. They are basically the same ones as given by Plummer [9]. The point is the existence of a set S of vertices satisfying the two conditions in the following lemma. Plummer called such a set a $\{e_1, e_2\}$*-blocker*:

Lemma 4. *Suppose that there is an independent pair of edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in a graph G of even order which does not extend to a perfect matching. Then G contains a set S of vertices such that:*

(i) $S \supset \{u_1, v_1, u_2, v_2\}$, *and*
(ii) $|S| \le o(G - S) + 2$,

where $o(H)$ stands for the number of odd components, that is, components of H each of which consists of an odd number of vertices.