Discrete Mathematics 310 (2010) 2519–2541

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Enumeration and limit laws of dissections on a cylinder

Juanjo Rué

Laboratorie d'Informatique, École Polytechnique, 91128 Palaiseau-Cedex, France

ARTICLE INFO

Article history:
Received 16 January 2010
Received in revised form 9 June 2010
Accepted 14 June 2010
Available online 5 July 2010

Keywords:
Enumerative combinatorics
Analytic combinatorics
Symbolic method
Map enumeration
Singularity analysis

ABSTRACT

We compute the generating function for triangulations on a cylinder, with the restriction that all vertices belong to its boundary and that the intersection of a pair of different faces is either empty, a vertex or an edge. We generalize these results to maps with either constant ($\{k\}$-dissections) or unrestricted (unrestricted dissections) face degree. We apply singularity analysis to the resulting generating functions to obtain asymptotic estimates for their coefficients, as well as limit distributions for natural parameters.

1. Introduction

The enumeration of triangulations on a labelled disc is one of the first non-trivial problems in enumerative combinatorics. This question gives rise to the well-known Catalan numbers, which appear in different contexts of discrete mathematics [14]. This counting problem can be generalized in the context of map enumeration as follows: let $\mathbb{S}$ be a surface with boundary. We say that a triangular map on $\mathbb{S}$ (i.e., faces have degree 3) is a *simplicial decomposition* if each face is incident with 3 vertices and the intersection of a pair of different faces is either empty, a vertex or an edge. One can easily check that simpliciality is equivalent to the non-existence of neither loops nor multiple edges. Under these assumptions, what is the number of simplicial decompositions of $\mathbb{S}$ with the restriction that all vertices lie on the boundary of $\mathbb{S}$? Observe that this general problem covers the enumeration of triangulations on a labelled disc, as a disc is a surface with boundary. For other surfaces with boundary some work has been done: the Möbius band was first studied in [3], and the picture was completed in [12] (see also [7,8]). In the present work we study the next step of this problem: the study of simplicial decompositions on a cylinder, without internal vertices.

Our techniques let us study general families of maps on a cylinder, with all vertices on the boundary. We say that a map on a cylinder is a *dissection* if each face of degree k is incident with exactly k vertices, and if the intersection of two different faces is either empty, a single vertex or an edge. In this work we obtain the enumeration of dissections on a cylinder where faces have degree k (also called $\{k\}$-*dissections*) and also the enumeration of dissections where the degree of each face is unrestricted (*unrestricted dissections*). In particular, simplicial decompositions are $\{3\}$-dissections. Examples of a simplicial decomposition, a $\{4\}$-dissection and an unrestricted dissection are shown in Fig. 1.

The main contribution of this paper is the method used to get this enumeration: in previous works the main tool was the decomposition induced by the root (combinatorial surgery arguments), a method which is reminiscent of the seminal works of Tutte on map enumeration [15,16]. In our study we introduce a composition scheme, which arises from a bijective characterization of the combinatorial families under study. These bijective techniques make the analysis more transparent than the one made using root decompositions (see [13] for a combinatorial study using bijective tools). In particular, we avoid the long inclusion–exclusion argument used in [12].

E-mail addresses: rue1982@lix.polytechnique.fr, juanjo.rue@gmail.com.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.06.023

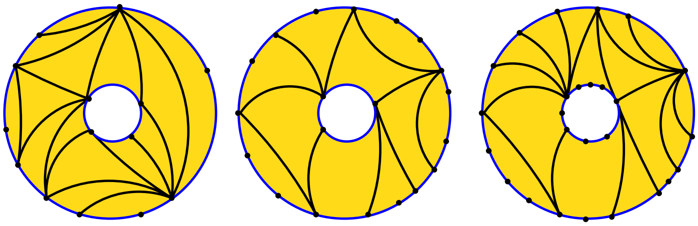

Fig. 1. A simplicial decomposition, a $\{4\}$-dissection and an unrestricted dissection.

Once we obtain the exact enumeration for this families using the associated generating functions, we get asymptotic estimates for the coefficients. Asymptotic results for families of rooted maps on arbitrary surfaces with boundary are obtained in [1]. In this work the authors consider a unique edge root on one of the boundary components. The main difference in the present work with respect to [1] is that in our analysis each boundary is rooted, hence the maps under study carry a pair of roots. The main analytic tool in this part is the transfer of singularities [5], which provides a systematic method to translate analytic properties of a generating function into asymptotic estimates of their coefficients. More concretely, let $h_n^{\{k+1\}}$ denote the number of $\{k+1\}$-dissections on a cylinder with n faces, such that all the vertices belong to the boundary. Our main result in this part states that

$$h_n^{\{k+1\}} \sim_{n\to\infty} \frac{(k-1)^2}{16} \cdot n \cdot \rho_{k+1}^{-n},$$

where $\rho_{k+1} = (k-1)^{k-1}/k^k$ is the radius of convergence of the generating function of $\{k+1\}$-dissections on a labelled disc. In particular, we observe that the exponential growth of the coefficients depends on the *type* of the dissection (that is, in the allowed degrees for the faces).

We also study parameters on a random $\{k\}$-dissection with a fixed number of vertices. The main technical problem in this part is that we cannot apply general theorems as the quasi-powers theorem [9]: we need to make a case-by-case analysis in order to find the limit distribution of the corresponding random variable. Using generating functions we are able to extract factorial moments of the random variables under study, from which we deduce by the method of moments the existence of a limiting distribution. Some additional work must be done in order to characterize this limit. In this point, the Laplace transform is the main tool in order to get the expression of the density probability function in terms of its factorial moments. More concretely, let $\mathbf{h}_n$ be a simplicial decomposition on a cylinder (with all its vertices on the boundary) chosen uniformly at random among all simplicial decompositions on a cylinder with n vertices. Denote by $\mathbf{Z}_n$ the size of the core of $\mathbf{h}_n$ (see Section 8.2 for a proper definition) and by $\mathbf{W}_n$ the number of vertices of $\mathbf{h}_n$ on one of the boundary components. Let $\mathbf{Z}, \mathbf{W}$ be random variables with density probability functions

$$f_{\mathbf{Z}}(t) = t \cdot \operatorname{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t), \qquad f_{\mathbf{W}}(t) = \frac{8}{\pi}\sqrt{t-t^2}\,\mathbb{I}_{[0,1]}(t).$$

Where $\mathbb{I}_{[0,\infty[}(t)$ (resp. $\mathbb{I}_{[0,1]}(t)$) is the characteristic function of the set $[0,\infty[$ (resp. $[0,1]$), and $\operatorname{erfc}(t)$ is the complementary error function. Our main result in this part states that $\mathbf{Z}_n/\sqrt{n} \to \mathbf{Z}$ and $\mathbf{W}_n/n \to \mathbf{W}$ in distribution.

To conclude we show that the framework presented in this paper explains previous related works. In particular, as a direct consequence of our results, we are able to generalize results of Gao et al. [8] on the exact enumeration of simplicial decompositions on a cylinder, with the difference that there exists a unique root edge on one of the boundary components.

Outline of the paper. We start in Section 2 recalling the necessary background and setting our terminology. We continue studying simplicial decompositions in Section 3. In order to obtain the enumeration of $\{k\}$-dissections and unrestricted dissections, we need to study a combinatorial class (fundamental dissections) which is introduced in Section 4. Ideas used in Section 3 are refined in Sections 5 and 6 in order to obtain the generating functions for $\{k\}$-dissections and unrestricted dissections. In Section 7 we study the asymptotic enumeration of these families. We use analytical tools to study the distribution of parameters in Section 8. In Appendix A we introduce technical lemmas related to the integration of formal power series, which are used in the preceding sections. In Appendix B we present results, without the proofs, on dissections on a cylinder with a single root. In our approach these results are simple consequences of the ideas developed in the previous sections.

2. Preliminaries

In this section we introduce the basic notions of the symbolic method, which provides a direct way to translate combinatorial conditions into equations. We introduce also the basic definitions in the map framework. We apply the symbolic method to get the enumeration of certain families of maps defined on a disc. To conclude, we introduce generalities about the notation used in the rest of the work.

2.1. The symbolic method

A common technique to deal with enumerative problems is the language of generating functions. We use the methodology introduced by Flajolet and Sedgewick in the context of *analytic combinatorics* (see [6]).