Discrete Mathematics 312 (2012) 803–807

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A new expression for matching polynomials ☆

F.M. Dong

Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616, Singapore

ARTICLE INFO

Article history:
Received 13 January 2011
Received in revised form 15 November 2011
Accepted 15 November 2011
Available online 7 January 2012

Keywords:
Graph
Matrix
Matching
Matching polynomial

ABSTRACT

Let G be an arbitrary simple graph. Godsil and Gutman in 1978 and Yan et al. in 2005 established different expressions for the matching polynomial $\mu(G, x)$ in terms of $\det(xI_n - H)$ for some families of matrices H. This paper improves their results and simplifies the computation of $\mu(G, x)$.

1. Introduction

In this paper we consider simple graphs (i.e., a graph with no loops and parallel edges) only. For any graph G, let $V(G)$, $E(G)$ and $v(G)$ be its vertex set, edge set and order (i.e., $v(G) = |V(G)|$), respectively. If it is not mentioned elsewhere in this paper, we always assume that G is a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_\epsilon\}$, where $\epsilon = |E|$. A *matching* of G is a subset M of E such that each vertex of G is incident with at most one edge in M. For any integer $k \geq 0$, let $\phi_k(G)$ denote the number of matchings M of G with $|M| = k$. It is clear that $\phi_0(G) = 1$ and $\phi_1(G) = |E|$. One form of matching polynomial is $\sum_{k\geq 0} \phi_k(G)x^k$ (see [1]). In this paper, we study another form of matching polynomial which is defined below:

$$\mu(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \phi_k(G) x^{n-2k}. \tag{1.1}$$

This polynomial is also called the acyclic polynomial (see [4]). Throughout this paper, this polynomial $\mu(G, x)$ will be referred to as the matching polynomial of G.

Godsil and Gutman [2] showed that

$$\mu(G, x) = 2^{-\epsilon} \sum_{w} \det(xI_n - A(w)), \tag{1.2}$$

where the summation ranges over all 2^ϵ distinct ϵ-tuples $w = (w_1, w_2, \ldots, w_\epsilon)$, $w_j \in \{1, -1\}$ and the matrix $A(w) = (a_{j,k})$ with the tuple $w = (w_1, w_2, \ldots, w_\epsilon)$ is defined as follows: $a_{j,k} = w_s$ if $v_j v_k$ is the edge e_s and $a_{j,k} = 0$ if $v_j v_k \notin E$ for all j, k. Yan et al. [7] obtained a similar result that

$$\mu(G, x) = 2^{-\epsilon} \sum_{G^e} \det(xI_n + iA(G^e)), \tag{1.3}$$

☆ Supported by NIE AcRf funding (RI 5/06 DFM) of Singapore.
E-mail address: fengming.dong@nie.edu.sg.

0012-365X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.11.019

where the sum ranges over all 2^ϵ orientations G^e of G, i is the complex number with $i^2 = -1$ (i will be used to denote this number throughout this paper) and $A(G^e) = (a_{j,k})$ is the matrix defined as follows: $a_{j,k} = 1$ if (v_j, v_k) is an arc in G^e, $a_{j,k} = -1$ if (v_k, v_j) is an arc in G^e and $a_{j,k} = 0$ otherwise.

This paper generalizes the above results by showing that if F is a subset of E such that every pair of cycles in $G-F$ (i.e., the subgraph obtained from G by removing all edges in F) are edge-disjoint, then

$$\mu(G,x) = 2^{-|F|}\sum_B \det(xI_n - B), \tag{1.4}$$

where the sum ranges over all matrices in a set of $2^{|F|}$ matrices $B = (b_{j,k})$ with the property that $b_{j,k} \times b_{k,j} = 1$ when $v_jv_k \in E$ and $b_{j,k} = b_{k,j} = 0$ otherwise (see Corollary 2.2). When $F = E$, this result implies (1.2) and (1.3).

2. Main result

For any graph G, let $\mathcal{M}(G)$ be the set of matrices $(a_{j,k})_{n\times n}$ such that $a_{j,k}a_{k,j} = 1$ if $v_jv_k \in E$ and $a_{j,k} = a_{k,j} = 0$ otherwise. Note that $(a_{j,k}) \in \mathcal{M}(G)$ is an adjacency matrix of G if $a_{j,k} = 1$ whenever $v_jv_k \in E$. It is well known (see [4–6]) that $\mu(G,x) = \det(xI_n - A)$ if G is a forest and A is an adjacency matrix of G. This result is actually a particular case of the following result due to Graovac and Polanksy [3].[1]

Theorem 2.1 ([3]). *Let G be a graph in which every pair of cycles are edge-disjoint and $A = (a_{j,k})$ be any matrix in $\mathcal{M}(G)$. Assume that for every cycle $C : v_{r_1}v_{r_2}\cdots v_{r_s}v_{r_1}$ in G, the following condition always holds:*

$$a_{r_1,r_2}, a_{r_2,r_3}, \ldots, a_{r_s,r_1} \in \{1,-1,i,-i\} \quad \textit{and} \quad a^2_{r_1,r_2}a^2_{r_2,r_3}\cdots a^2_{r_s,r_1} = -1.$$

Then $\mu(G,x) = \det(xI_n - A)$. □

By Theorem 2.1, if G is a forest, then $\mu(G,x) = \det(xI_n - A)$ holds for every matrix $A \in \mathcal{M}(G)$.

For $G = (V,E)$, where $V = \{v_1, v_2, \ldots, v_n\}$, assign every $e \in E$ a non-zero complex number w_e. We call $\{w_e\}_{e\in E}$ *the weight-function* of E, denoted by $\mathbf{w}_G$ (or simply by $\mathbf{w}$). Let $\mathcal{M}(G,\mathbf{w})$ be the set of $(n\times n)$-matrices $(a_{j,k})$ satisfying the condition below:

$$\begin{cases} a_{j,k} \in \{w_e, -w_e\}, & \text{if } j<k \text{ and } v_jv_k = e \in E;\\ a_{j,k} = 1/a_{k,j}, & \text{if } j>k \text{ and } v_jv_k \in E;\\ a_{j,k} = 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

Note that $\mathcal{M}(G,\mathbf{w})$ contains exactly $2^{|E|}$ matrices and $\mathcal{M}(G,\mathbf{w}) \subseteq \mathcal{M}(G)$.

By the notation of a weight-function $\mathbf{w} = \{w_e\}_{e\in E}$, the result of (1.2) due to Godsil and Gutman [2] is equivalent to the expression below with $w_e = 1$ for all $e \in E$:

$$\mu(G,x) = 2^{-|E|}\sum_{A\in\mathcal{M}(G,\mathbf{w})} \det(xI_n - A). \tag{2.2}$$

The result of (1.3) due to Yan et al. [7] is also equivalent to (2.2) with $w_e = i$ for all $e \in E$. We shall show that (2.2) actually always holds as long as $w_e \neq 0$ for all $e \in E$.

Let $G = (V,E)$ be any graph with $V = \{v_1, v_2, \ldots, v_n\}$ and weight-function $\mathbf{w} = \{w_e\}_{e\in E}$, F be a subset of E and $A = (a_{j,k})$ be an $n\times n$ matrix with $a_{j,k} \neq 0$, whenever $v_jv_k \in E$. Let

$$\mathcal{G}_F(G) = \{G - F - \{v_j, v_k : v_jv_k \in F'\} : F' \subseteq F\}, \tag{2.3}$$

where $G-F-V'$ is the subgraph of $G-F$ after deleting all vertices in V', and $\mathcal{M}_F(A)$ be the set of matrices $(d_{j,k})_{n\times n}$ satisfying the following condition:

$$\begin{cases} d_{j,k} \in \{a_{j,k}, -a_{j,k}\}, & \text{if } j<k \text{ and } v_jv_k \in F;\\ d_{j,k} = 1/d_{k,j}, & \text{if } k<j \text{ and } v_jv_k \in F;\\ d_{j,k} = a_{j,k}, & \text{otherwise.}\end{cases} \tag{2.4}$$

Note that $G - F \in \mathcal{G}_F(G)$ and every graph of $\mathcal{G}_F(G)$ is a subgraph of $G-F$. It is also clear that $|\mathcal{M}_F(A)| = 2^{|F|}$. Note that if $A \in \mathcal{M}(G)$, then $A \in \mathcal{M}_F(A) \subseteq \mathcal{M}(G)$ for any $F \subseteq E$.

For any $n\times n$ matrix $A = (a_{j,k})$ and any non-empty subset I of $\{1,2,\ldots,n\}$, let $A[I]$ be the matrix obtained from A by removing rows $s_1, s_2, \ldots, s_r$ and columns $s_1, s_2, \ldots, s_r$, where $\{s_1, s_2, \ldots, s_r\} = \{1,2,\ldots,n\} - I$.

[1] This result was explained in [3]. It may have also appeared in some other articles.