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Six signed Petersen graphs, and their automorphisms

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ABSTRACT

Up to switching isomorphism, there are six ways to put signs on the edges of the Petersen graph. We prove this by computing switching invariants, especially frustration indices and frustration numbers, switching automorphism groups, chromatic numbers, and numbers of proper 1-colorations, thereby illustrating some of the ideas and methods of signed graph theory. We also calculate automorphism groups and clusterability indices, which are not invariant under switching. In the process, we develop new properties of signed graphs, especially of their switching automorphism groups.

1. Introduction

The Petersen graph P is a famous example and counterexample in graph theory, making it an appropriate subject for a book (see [11]). With signed edges, it makes a fascinating example of many aspects of signed graph theory as well. There are 2^{15} ways to put signs on the edges of P, but in many respects only six of them are essentially different. We show how and why that is true as we develop basic properties of these six signed Petersens.

The fundamental property of signed graphs is balance. A signed graph is *balanced* if all its circles (circuits, cycles, polygons) have positive sign product. Harary introduced signed graphs and balance [9] (though they were implicit in [12, Section X.3]). Cartwright and Harary used them to model social stress in small groups of people in social psychology [6]. Subsequently, signed graphs have turned out to be valuable in many other areas, some of which we shall allude to in what follows.

The opposite of balance is frustration. Most signatures of a graph are unbalanced; but they can be made balanced by deleting (or, equivalently, negating) edges. The smallest number of edges whose deletion makes the graph balanced is the *frustration index*, a number which is implicated in certain questions of social psychology [1,10] and spin-glass physics [16,3]. We find the frustration indices of all signed Petersen graphs (Theorem 7.2).

The second basic property of signed graphs is switching equivalence. Switching is a way of turning one signature of a graph into another, without changing circle signs. Many properties of signed graphs are unaltered by switching, the frustration index being a notable example. The first of our main theorems is that there are exactly six equivalence classes of signatures of P under the combination of switching and isomorphism (Theorem 5.1). Fig. 1.2 shows a representative of each switching isomorphism class. In each representative, the negative edges form a smallest set whose deletion makes the signed Petersen balanced. Hence, we call them *minimal signatures* of P (see Theorem 7.2). Because there are only six switching isomorphism classes of signatures, the frustration index of every signature of P can be found from those of the minimal signatures.

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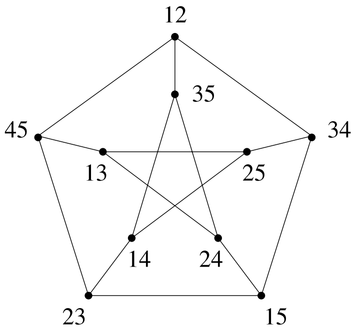

Fig. 1.1. *P*, the Petersen graph.

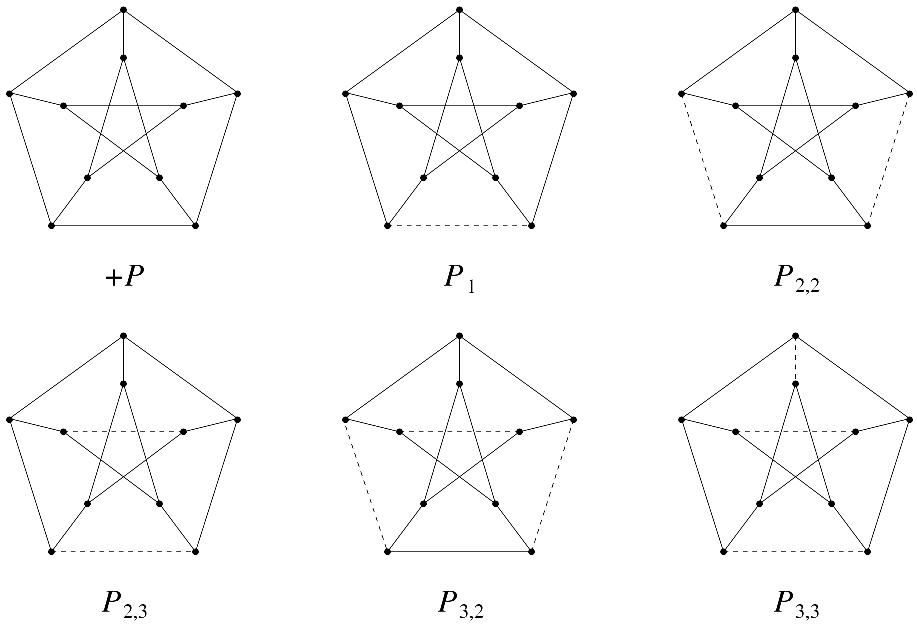

Fig. 1.2. The six switching isomorphism types of signed Petersen graph. Solid lines are positive; dashed lines are negative.

The second main theorem, which occupies the bulk of this paper, is a computation of the automorphism and switching automorphism groups of the six minimal signatures (Theorem 8.12). An automorphism has the obvious definition: it is a graph automorphism that preserves edge signs. This group is not invariant under switching. It is not even truly signed-graphic, since in what concerns automorphisms a signed graph is merely an edge 2-colored graph. The proper question for signed graphs regards the combination of switching with an automorphism of the underlying graph. The group of switching automorphisms of a signed graph is, by its definition, invariant under switching, so just six groups are needed to know them all. Some of the groups are trivial, but one is so complicated that it takes pages to describe it thoroughly.

Isomorphic minimal signatures may not be equivalent under the action of the switching group. The number of switching-inequivalent signatures of a given minimal isomorphism type is deducible from the order of the switching automorphism group (Section 8.3).

Two further properties are treated more concisely. First, a signed graph can be colored by signed colors. This leads to two chromatic numbers, depending on whether or not the intrinsically signless color 0 is accepted. The chromatic numbers are invariant under switching (and isomorphism); thus they help to distinguish the six minimal signatures by showing their inequivalence under switching isomorphism (Theorem 9.2). The two chromatic numbers are aspects of two chromatic polynomials, but we make no attempt to compute those polynomials, as they have degree 10.

Finally, we take a brief excursion into a natural generalization of balance called *clusterability* (Section 10). This, like the automorphism group, is not switching invariant, but it has attracted considerable interest, most recently in connection with the organization of data (see [2]), and has complex properties that have been but lightly explored.

Signed graphs, signed Petersen graphs in particular, have other intriguing aspects that we do not treat. Two are mentioned in the concluding section, but they hardly exhaust the possibilities.