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Linear codes arising from the Gale transform of distinguished subsets of some projective spaces

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ABSTRACT

Applying the Gale transform on certain linear and non-linear geometrical objects, and studying the orbits under the action of the associated automorphism groups in the higher-dimensional space, we construct some families of cap codes and other structures admitting the same automorphism groups.

1. Introduction

The Gale transform of a set $\mathcal{T}$ consisting of γ labelled points of a projective space $\mathrm{PG}(r, q)$ is an involution, defined up to automorphisms, which maps $\mathcal{T}$ into a set $\mathcal{T}'$ consisting of γ labelled points of $\mathrm{PG}(s, q)$ with $\gamma = r + s + 2$.

The simplest way to define the Gale transform of a set of points is in terms of projective coordinates. Choose homogeneous coordinates in such a way that the coordinates of the points of $\mathcal{T}$ are the rows of the matrix

$$\begin{pmatrix} I_{r+1} \\ A \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix and A is an $(s + 1) \times (r + 1)$ matrix. Then, the Gale transform of $\mathcal{T}$ is the set $\mathcal{T}'$ consisting of the points of $\mathrm{PG}(s, q)$ whose homogeneous coordinates are the rows of the matrix

$$\begin{pmatrix} {}^{T}\!A \\ I_{s+1} \end{pmatrix},$$

where ${}^{T}\!A$ is the transpose matrix of A.

The Gale transform has many geometrical and group-theoretical properties which make it a valuable tool in several disciplines such as optimization, coding theory and algebraic geometry. Here we list some recent results that will be useful. The interested reader is referred to [3,4] for proofs of the results and further details on the Gale transform.

Lemma 1.1. *If ℓ is a line in some projective space $\mathrm{PG}(r, q)$ and $\mathcal{T} \subseteq \ell$, with $|\mathcal{T}| = r + s + 2$, then the Gale transform $\mathcal{T}'$ of $\mathcal{T}$ is contained in the unique normal rational curve of $\mathrm{PG}(s, q)$ containing the fundamental frame.*

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Lemma 1.2. *The Gale transform of a k-cap in a projective space* $\mathrm{PG}(r,q)$, $k \geq r+4$, *is a k-cap in* $\mathrm{PG}(k-r-2,q)$.

Lemmas 1.1 and 1.2 together yield the following crucial result.

Theorem 1.3. *Let* $\mathcal{T}$ *be any set consisting of k points in* $\mathrm{PG}(r,q)$, $r \geq 2$ *and* $k \geq r+4$. *Then the Gale transform* $\mathcal{T}'$ *of* $\mathcal{T}$ *is a k-cap in* $\mathrm{PG}(k-r-2,q)$.

When constructing an error correcting code, it is convenient to have some control on its automorphism group. The following result generalizing [3, Proposition 2.5] provides a useful tool for this task.

Theorem 1.4. *Let* $\mathcal{T}$ *be any subset of* $\mathrm{PG}(r,q)$ *consisting of at least* $k=r+4$ *points and* $\mathcal{T}'$ *its Gale transform. Then* $\mathcal{T}$ *and* $\mathcal{T}'$ *have isomorphic collineation groups.*

Proof. Let $\mathcal{T}=\{p_1,\ldots,p_k\}$. The $(r+1)\times k$ matrix $(p_1\ldots p_k)$ determines a vector subspace $V=V(r+1,q)$ in the vector space $V(k,q)$. This is due to the fact that $\mathcal{T}$ always contains the fundamental frame of $\mathrm{PG}(r,q)$. The Gale transform of $\mathcal{T}$ produces a new set in $\mathrm{PG}(k-r-2,q)$ containing the fundamental frame which generates the orthogonal complement $V^{\perp}$ of V in $V(k,q)$. The theorem now follows from [11, Theorem 2.1]. $\square$

The main idea in this paper is to use the Gale transform of some remarkable objects in projective spaces to construct other interesting objects and possibly study the associated linear codes.

An interesting question that can be posed is the following. When does it happen that the Gale transform of a subset of the projective space is embedded in a Veronese variety? This fact is relevant because a Veronese variety is always a cap and its extensions could give rise to interesting cap codes. In this paper we have studied the cases of a classical unital in a projective plane of order $q=4$, a distinguished 1-set in $\mathrm{PG}(3,4)$, a Baer subplane in $\mathrm{PG}(2,4)$ and a Baer subplane in $\mathrm{PG}(2,9)$.

2. The Gale transform of $\mathcal{H}(2,4)$ and its code

A classical unital $\mathcal{H}(2,q)$ in a projective plane $\mathrm{PG}(2,q)$ of square order q is the set of all absolute points of a non-degenerate Hermitian form. It consists of $q\sqrt{q}+1$ points such that each line meets $\mathcal{H}(2,q)$ at either 1 (tangent) or $\sqrt{q}+1$ points (secant). The collineation group preserving $\mathcal{H}(2,q)$ in $\mathrm{PGL}(3,q)$ is the subgroup $\mathrm{PGU}(3,q)$, see [8] and also [9, Theorem 2.50] or [6, Example A.9].

From now on we shall focus on the projective plane $\mathrm{PG}(2,4)$ over the field $\mathrm{GF}(4)=\{0,1,\omega,\omega^2\}$, with $\omega^2+\omega+1=0$.

The projective plane $\mathrm{PG}(2,4)$ contains exactly 280 unitals, 18 of which pass through the fundamental points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Each of these $\mathcal{H}(2,4)$'s admits one tangent and four secants through each of its points. Assume that $\mathcal{H}=\mathcal{H}(2,4)$ has equation

$$\omega(XY^2+XZ^2+YZ^2)+\omega^2(X^2Y+X^2Z+Y^2Z)=0.$$

A quick computation [2] provides the point set of $\mathcal{H}$, as in Table 5. Its Gale transform is the set $\mathcal{H}'$ – whose points constitute a 9-cap in $\mathrm{PG}(5,4)$, see Table 6 – admitting $\mathrm{PGU}(3,4)$ as automorphism group.

Actually, the group $G=\mathrm{PGU}(3,4)$ acting on $\mathcal{H}'$ has, among several others, the following orbits on points of $\mathrm{PG}(5,4)$:

- the point set $\mathcal{H}'$ itself;
- two orbits, say $\mathcal{R}$ and $\mathcal{T}$, of size 12;
- an orbit $\mathcal{S}$ of size 9.

It turns out that one of the orbits of size 12, say $\mathcal{R}$, is a cap in $\mathrm{PG}(5,4)$ that together with $\mathcal{H}'$ gives rise to a 21-cap $\mathcal{V}$ (see Tables 6 and 7). The other orbit $\mathcal{T}$ of size 12, together with the other orbit $\mathcal{S}$ of size 9, gives rise to a plane, say π. It turns out that G is reducible in its action on the points of $\mathrm{PG}(5,4)$ (see Tables 8 and 9). Looking at the list of maximal subgroups of $\mathrm{PSL}(6,q)$ as presented by Kleidman [10], we conclude that the stabilizer of the 21-cap $\mathcal{V}$ is the symmetric square of $\mathrm{PSL}(3,4)$, see

[1, Proposition 4]; hence $\mathcal{V}$ is a Veronese surface with nucleus π, see [7, Chapter 25]. In Table 10 we have reported the 21 conics embedded in $\mathcal{V}$.

The plane π admits a regular hyperoval $\mathcal{C}$ consisting of six points and preserved by the group $A_6\cong\mathrm{PSL}(2,9)$, see [12]. For instance, such a hyperoval in π is given by

$$\begin{aligned}\mathcal{C}=\{&(1,0,0,0,\omega,\omega^2),(0,1,\omega,0,0,\omega),(0,0,1,\omega^2,1,\omega^2),\\&(1,\omega,\omega,\omega^2,\omega^2,\omega^2),(1,\omega^2,\omega,\omega,1,0),(1,1,0,1,0,0)\},\end{aligned}$$

and the union $\mathcal{V}\cup\mathcal{C}$ is a non-complete 27-cap of $\mathrm{PG}(5,4)$ whose stabilizer is the subgroup A_6 of the group $\mathrm{PGU}(3,4)$ of $\mathcal{V}$.

We checked with MAGMA that no hyperplane of $\mathrm{PG}(5,4)$ is disjoint from $\mathcal{V}\cup\mathcal{C}$. Hence, $\mathcal{V}\cup\mathcal{C}$ gives rise to a cap code which is a linear $[27,6,16]_4$-code admitting $C_3\times A_6$ as its automorphism group, and the weight distribution of this code is

$$(0,1),(16,513),(18,180),(20,2160),(22,864),(24,270),(26,108).$$