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The Lemma of Tangents reformulated

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ABSTRACT

The famous Lemma of Tangents describes a useful (algebraic) relation between the tangents through three points of an arc in a Desarguesian projective plane. Because the formulation of the lemma assumes the three points to have coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, it is sometimes not so evident to apply when studying arc subsets of more than three points.

In this paper, we reformulate the Lemma of Tangents in a concise way which is independent of the chosen basis of the projective plane. We also express the consequences of this lemma for sets of more than three arc points in the form of linear equations. To show that our framework is helpful we provide a new and direct proof of the fact that every q-arc in $PG(2, q)$ must be part of a conic when q is odd.

1. Introduction

Consider the Desarguesian projective plane $PG(2, q)$ over a finite field $K = GF(2, q)$. A k-arc $\mathcal{K}$ of $PG(2, q)$ is a set of k points of $PG(2, q)$ such that no three of them are collinear. A line is called an *external line*, *tangent* or *secant* to $\mathcal{K}$ if it intersects $\mathcal{K}$ in 0, 1 or 2 points respectively. Of the $q + 1$ lines through a point of $\mathcal{K}$, $k - 1$ are secants and $t = q + 2 - k$ are tangents.

We will be especially interested in the case where t is small. When $t = 0$, i.e., when the arc has size $q + 2$, it is called a *hyperoval*. Hyperovals only exist when q is even. In this text we shall mostly restrict ourselves to the case where q is odd and hyperovals will not be considered.

For $t = 1$ the standard example of a $q + 1$-arc is provided by the points of a conic. A celebrated result of Segre [4,5] proves that, when q is odd, every $q + 1$-arc is necessarily of this type. This result is a consequence of the following lemma.

Lemma 1 (*Lemma of Tangents*)**.** *In the Desarguesian projective plane of order q, consider an arc $\mathcal{K}$ of size $q + 2 - t$ which contains the three points $e_1(1, 0, 0)$, $e_2(0, 1, 0)$ and $e_3(0, 0, 1)$. Then*

$$\prod_{i=1}^{t} A_i B_i C_i = -1,$$

where $A_i, B_i, C_i,\ i = 1, \ldots, t$, denote the (non-zero) coefficients in the equations

$$y - A_i z = 0, \qquad (z - B_i x = 0, \qquad x - C_i y = 0,\ \text{resp.})$$

of the tangents to $\mathcal{K}$ through e_1 (e_2, e_3, resp.).

For a proof of this well-known lemma we refer to [3]. Note that the lemma is valid also when q is even.

The specific choice of the coordinates of e_1, e_2, e_3 does not make the Lemma of Tangents less general. Indeed, for any three different points of any given arc $\mathcal{K}$ we can always find a projectivity that maps these points to e_1, e_2 and e_3, and this projectivity will map $\mathcal{K}$ onto a set $\mathcal{K}'$ which is again an arc.

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However, this specific formulation is less useful if we intend to consider properties of the tangents of more than three points of $\mathcal{K}$. In Section 2 we obtain a different (and in our opinion, elegant) formulation of the Lemma of Tangents in which the points e_1, e_2 and e_3 do not play a special role (Lemma 2). In Section 3 we introduce some further concepts to enable us to apply Lemma 2 more easily in terms of coordinates, and in Section 4 we use these to establish algebraic identities (10) and (11) which must be satisfied by all points and tangents of a given arc. Finally in Section 5 we use these identities to tackle the case when q is odd and $t = 2$.

After a first version of this paper was submitted to the journal, it was brought to our attention that also Simeon Ball had recently formulated a coordinate free version of the Lemma of Tangents [1]. His reformulation is similar to our Lemma 2 but is stated in terms of polynomials. We have adapted Section 3 so that it fits his ideas and terminology more closely. From Section 4 onwards there is little or no overlap between his paper and ours.

Segre [6] already proved that in that case all q-arcs must necessarily consist of all points of a conic, except one. (The original proof contained a mistake and was amended by Büke [2].) Later Thas [7] provided a different proof of the same fact using algebraic geometric arguments applied to the dual of the arc (where the tangents are interpreted as points on an algebraic curve). Both proofs also make use of the following combinatorial result on arcs of size q: there must exist at least one point outside the arc which is incident with at least five tangents to that arc.

In Section 5 we shall give yet another proof based on the identities developed earlier (cf. Theorem 1). We claim that our proof is more direct than the other two. It also does not need the combinatorial property cited above. At the least it establishes the validity of our techniques.

2. Reformulation of the lemma

We henceforth assume that the arc $\mathcal{K}$ contains the two points a and b with homogeneous coordinates $a(1, 0, 0)$ and $b(0, 1, 0)$. This we can do without loss of generality.[1]

As a consequence, any further point $r(x, y, z)$ of $\mathcal{K}$ must have $z \neq 0$, and hence we can *normalize* its coordinate triple to the form $(x, y, 1)$ by dividing each coordinate by z. In other words, all points of $\mathcal{K}$ except a and b belong to the affine plane $\mathrm{AG}(2, q)$ obtained from $\mathrm{PG}(2, q)$ by removing the line ab at infinity with equation $z = 0$. In what follows we shall always assume that point coordinates are normalized in this way, unless explicitly indicated otherwise. (The triples $(1, 0, 0)$ and $(0, 1, 0)$ will also be considered normalized.)

The line with equation $Lx + My + Nz = 0$ has coordinates (L, M, N) which we shall likewise expect to be normalized in the following way: if $M \neq 0$ then $M = 1$, otherwise if $L \neq 0$ then $L = 1$, otherwise $N = 1$. In other words, a line will have an equation of the form $y = -Lx - Nz$, $x = -Nz$ or $z = 0$.

Let r be a point with normalized coordinates (x, y, z) and R a line with normalized coordinates (L, M, N). We define

$$rR \stackrel{\text{def}}{=} Lx + My + Nz, \tag{1}$$

i.e., the dot product of the normalized coordinate triples of r and R. Clearly $rR = 0$ if and only if r lies on R.

Define a *t-fan* of the plane (or simply, a *fan*, if t is clear from context) to be a pair $\mathbf{r} = (r, \{R_1, \ldots, R_t\})$ where r is a point and $R_1, \ldots, R_t$ are different lines through r. (The point r is called the *center* of the fan.) A fan is called *affine* if and only if its center is affine.

A *tangent* fan of the arc $\mathcal{K}$ consists of a point r of $\mathcal{K}$ together with the t tangents to $\mathcal{K}$ through r. The tangent fans through a and b will be denoted by $\mathbf{a}$ and $\mathbf{b}$ respectively. Except for these two, all tangent fans of $\mathcal{K}$ are affine.

A fan $\mathbf{r} = (r, \{R_1, \ldots, R_t\})$ will be called *disjoint* from a point s if and only if s does not lie on any of the lines $R_1, \ldots, R_t$. Two fans $\mathbf{r}$ and $\mathbf{s}$ will be called disjoint if and only if each fan is disjoint from the center of the other. By definition, all tangent fans of $\mathcal{K}$ are disjoint.

Consider t-fans $\mathbf{r} = (r, \{R_1, \ldots, R_t\})$ and $\mathbf{s} = (s, \{S_1, \ldots, S_t\})$. If $\mathbf{r}$ and $\mathbf{s}$ are disjoint, we define the *ratio* $(\mathbf{r}/\mathbf{s})$ as follows

$$\left(\frac{\mathbf{r}}{\mathbf{s}}\right) \stackrel{\text{def}}{=} (-1)^{t+1} \frac{(rS_1)(rS_2)\cdots(rS_t)}{(sR_1)(sR_2)\cdots(sR_t)}. \tag{2}$$

Because $\mathbf{r}$ and $\mathbf{s}$ are disjoint, the ratio is always different from 0 and ∞.

The definition of ratio clearly depends on the way we chose to normalize point and line coordinates: a different normalization will yield different ratios. However, a product of ratios in which every fan occurs the same number of times in the numerator as in the denominator, will be projectively invariant. In particular, for two fans $\mathbf{r}$ and $\mathbf{s}$ we always find that $(\mathbf{r}/\mathbf{s})(\mathbf{s}/\mathbf{r}) = 1$.

We are now ready to reformulate the Lemma of Tangents:

[1] This somewhat breaks symmetry, which seems to contradict what we stated in the introduction. This assumption will however disappear in the formulation of our Lemma 2 and mainly serves to make computations simpler. In later sections we will need a and b to belong to $\mathcal{K}$, but then all other points of the arc will still be interchangeable.