Discrete Mathematics 310 (2010) 1507–1519

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Absolute retracts and varieties generated by chordal graphs☆

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ARTICLE INFO

Article history:
Received 26 March 2007
Received in revised form 18 August 2009
Accepted 25 August 2009
Available online 5 March 2010

Keywords:
Retraction
Absolute retract
Variety
Chordal graphs
Monophonic convexity
Hole
Hole base

ABSTRACT

Graphs that are retracts of each supergraph in which they are isometric are called absolute retracts with respect to isometry, and their structure is well understood; for instance, in terms of building blocks (paths) and operations (products and retractions). We investigate the larger class of graphs that are retracts of each supergraph in which all of their holes are left unfilled. These are the absolute retracts with respect to holes, and we investigate their structure in terms of the same operations of products and retractions. We focus on a particular kind of hole (called a stretched hole), and describe a class of simple building blocks of the corresponding absolute retracts. Surprisingly, these also turn out to be precisely those absolute retracts that can be built from chordal graphs. Monophonic convexity is used to analyse holes on chordal graphs.

1. Definitions and background

We shall assume that all graphs are finite, connected, and *reflexive*, i.e., there is a loop at each vertex; these loops are omitted in all figures.

A *homomorphism* of a graph G to a graph H is mapping ϕ of $V(G)$ to $V(H)$ such that $\phi(g)\phi(g') \in E(H)$ whenever $gg' \in E(G)$. If there exists a homomorphism η of H to G and a homomorphism ϕ of G to H such that $\phi(\eta(h)) = h$ for all vertices of H, then ϕ is called a *retraction* and H is called a *retract* of G. This implies that some subgraph of G is an isomorphic copy of H; for this reason, we will assume that if a graph H is a retract of graph G, then H is a subgraph of G.

Let $G_1, G_2, \dots, G_k$ be graphs. The (categorical) *product* of the graphs is denoted by $G_1 \times G_2 \times \cdots \times G_k$, or $\prod_{i=1}^{k} G_i$, and it has vertex set $\{(x_1, x_2, \dots, x_k) \mid x_i \in V(G_i)\}$ and edge set $\{(x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k) \mid x_iy_i \in E(G_i)\}$. Note that as the graphs are reflexive, it is possible that $x_i = y_i$ for some i. If $G = G_1 = G_2 = \cdots = G_k$, we write G^k for $\prod_{i=1}^{k} G_i$.

A class of graphs that is closed under taking products and retracts is called a *variety*. In other words, a class of graphs $\mathcal{W}$ is a variety if the following two conditions are satisfied:

(i) If $H_1, H_2 \in \mathcal{W}$ then $H_1 \times H_2 \in \mathcal{W}$.
(ii) If G is in $\mathcal{W}$ and H is a retract of G then H is in $\mathcal{W}$.

For H to be a retract of G, certain necessary conditions must be fulfilled. We say that a graph H is an *isometric* subgraph of a graph G if for all vertices x and y of H, $d_H(x, y) = d_G(x, y)$ where $d_G(x, y)$ is the shortest path distance in G. Note that if H is a retract of G then it must be isometric in G. We say that H is an *absolute retract with respect to isometry* if H is a retract of graph G whenever H is an isometric subgraph of G. Denote the class of absolute retracts with respect to isometry by $\mathcal{AR}_I$. Isometry sometimes does not suffice; see for example Fig. 1. There are stronger necessary conditions, one of which will be the focus of this paper.

☆ Part of this paper was written while visiting the Royal Military College of Canada; the support of NSERC's grant 26535 is gratefully acknowledged. Part of this paper was written with support from Pavol Hell's NSERC Discovery Grant.

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doi:10.1016/j.disc.2009.08.013

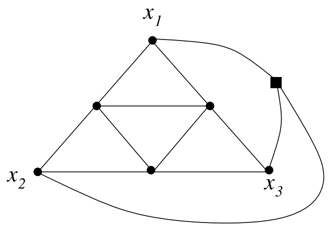

Fig. 1. Let A be the subgraph induced by the round vertices and let Q the whole graph. Then A is an isometric subgraph of Q, but not a retract of Q.

A *distance constraint* on a graph H is a function f with domain $D_f \subseteq V(H)$ whose values are non-negative integers. A *complete filler* of the distance constraint f is a vertex a such that $d(a, x) \leq f(x)$ for all $x \in D_f$. The set of all complete fillers of f is denoted by $F_H(f)$ (or just $F(f)$ if H is clear from the context). Given a vertex x of H and a non-negative integer k, the *disc of radius k centred at x* is the set $D_H(x, k) = \{y \in V(H) \mid d_H(x, y) \leq k\}$. Thus

$$F_H(f) = \bigcap_{x \in D_f} D_H(x, f(x)).$$

We also need to consider vertices that 'almost' satisfy a distance constraint f on a graph H. If $z \in D_f$, then a *z-relaxed filler of f* is a vertex a such that $d(x, a) \leq f(x)$ for all $x \in D_f \setminus \{z\}$.

A distance constraint f on a graph H is *feasible* if $F(f) \neq \emptyset$, and f is *infeasible* otherwise. We can define a partial order of distance constraints on a given graph H as follows: for two distance constraints f and f' on H we say that $f' \leq f$ exactly when $D_{f'} \subseteq D_f$ and $f'(x) \geq f(x)$ for all $x \in D_{f'}$. Moreover, we write $f' < f$ if $f' \leq f$ and $f' \neq f$. In particular, if f_x is the distance constraint with $D_{f_x} = D_f$ defined by

$$f_x(z) = \begin{cases} f(z) + 1 & \text{if } z = x \\ f(z) & \text{otherwise,} \end{cases}$$

then $f_x < f$. Additionally, if f' is a distance constraint on H where $D_{f'}$ is a proper subset of D_f and $f'(x) = f(x)$ for all $x \in D_{f'}$, then $f' < f$.

Proposition 1. *Let H be a graph and let f a distance constraint on H. For any distance constraint f' on H,*

(i) *if $f' \leq f$, then $F(f) \subseteq F(f')$.*
(ii) *if $f' < f$, then there exists a vertex $x \in D_f$ such that $f' \leq f_x < f$.*

Let H be a graph and let f be a distance constraint on H. We say that f is a *hole* if f minimally infeasible; that is f is an infeasible distance constraint on H and all distance constraints f' on H such that $f' < f$ are feasible. In particular, for any hole f on H, f_x is feasible for all $x \in D_f$. The cardinality of D_f is called the *size* of the hole f. We call a hole of size k a *degenerate hole* if $k = 2$ and a *non-degenerate hole* otherwise.

Let H be a graph, and let G be a supergraph of H. Any distance constraint on H is also a distance constraint on G. We say that *G fills a hole f* on H if there exists a vertex a in G such that a is in $F_G(f)$. If H is a retract of G, then any infeasible distance constraint on H must also be infeasible on G [23]. In particular, G cannot fill any hole on H. Consider the graphs A and Q in Fig. 1: The distance constraint f' with $D_{f'} = \{x_1, x_2, x_3\}$ and $f'(x_i) = 1$, $i = 1, 2, 3$ is a hole on A that is feasible on Q as the square vertex is in $F_Q(f')$. We say that H is an *absolute retract with respect to holes* if H is retract of a supergraph G whenever each hole on H is a hole on G. Denote the class of absolute retracts with respect to holes by $\mathcal{AR}_H$. Informally, H is in $\mathcal{AR}_H$ if H is retract of a supergraph G whenever G fills no holes of H. If f is a hole on H with $D_f = \{x_1, x_2\}$ then $d_H(x_1, x_2) = f(x_1) + f(x_2) + 1$ (see Proposition 4). A supergraph G of H filling a degenerate hole of H corresponds exactly to H not being isometric in G. Hence a supergraph G of H fills a hole of this sort if and only if H is not isometric in G. This shows that absolute retracts with respect to holes are a natural generalisation of absolute retracts with respect to isometry. However not all graphs are absolute retracts with respect to holes. The graph A of Fig. 1 is an example graph in $\mathcal{AR}_H$ but not in $\mathcal{AR}_I$, and the graph A of Fig. 2 is an example of graph not in $\mathcal{AR}_H$ [26].

As we will see later (Proposition 4), if f is a non-degenerate hole on a graph H, then $d(x, y) \leq f(x) + f(y)$ for all $x, y \in D_f$ and if f is a degenerate hole on H with $D_f = \{x, y\}$, then $d(x, y) = f(x) + f(y) + 1$. Thus we distinguish between holes f for which there exists $x, y \in D_f$ such that $d(x, y) < f(x) + f(y)$ and holes f for which such pairs of vertices in D_f do not exist; the former are called *squished holes* and the later are called *stretched holes*. The graph H is called *squished* if it has a squished hole and H is called *stretched* otherwise. Call a graph H an *absolute retract with respect to stretched holes* if H is a retract of a supergraph G whenever each stretched hole on H is a hole on G; denote the class of these graphs by $\mathcal{AR}_{SH}$.

A vertex x in a graph G is called *simplicial* if its neighbourhood is a clique in G; note that x is in its own neighbourhood as G is reflexive. For this reason, we call the neighbours of x other than itself the *nontrivial neighbours* of x. If there is a vertex y in G such that all neighbours of x are neighbours of y, then x is called *dismantlable*. Let $x_1, x_2, \ldots, x_n$ be an ordering of all vertices of G. The ordering is called a *perfect elimination ordering* if x_i is simplicial in $G_i = G \setminus \{x_1, x_2, \ldots, x_{i-1}\}$ for $i = 1, 2, \ldots, n - 1$