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Hamiltonicity in vertex envelopes of plane cubic graphs

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Tony Hilton has been our good friend and colleague for many years. His work as a mathematician has been both broad and deep, including work on hamiltonian cycles ([14,5,15], among others). For his wonderful work in mathematics, for his friendship and for his interest in hamiltonian cycles, we dedicate this paper to Tony.

Keywords:
Hamiltonian
Plane graph
Cubic graph
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ABSTRACT

In this paper we study a graph operation which produces what we call the "vertex envelope" $G^*{}_V$ from a graph G. We apply it to plane cubic graphs and investigate the hamiltonicity of the resulting graphs, which are also cubic. To this end, we prove a result giving a necessary and sufficient condition for the existence of hamiltonian cycles in the vertex envelopes of plane cubic graphs. We then use these conditions to identify graphs or classes of graphs whose vertex envelopes are either all hamiltonian or all non-hamiltonian, paying special attention to bipartite graphs. We also show that deciding if a vertex envelope is hamiltonian is NP-complete, and we provide a polynomial algorithm for deciding if a given cubic plane graph is a vertex envelope.

1. Introduction

We use the notation and terminology of West [19]. The *hamiltonicity* of a graph G is the answer to the question, "Does G contain a hamiltonian cycle?" Although many classes of graphs have been shown to be hamiltonian (for example, 4-connected plane graphs [18], graphs with high degree [6], and graphs with complete closures [3]), the general problem of hamiltonicity remains and is NP-complete. The variety of attacks on the hamiltonicity question includes studying graphs $f(G)$ derived from graphs G in terms of structures to be found in the graphs G. Examples of this approach include finding dominating cycles in G to show that the line graph of G is hamiltonian [13], showing that if G is 2-connected, then its square is hamiltonian [7,8], and showing that finding an EPS subgraph in G assures that its total graph is hamiltonian [10].

In this paper, we continue this tradition by examining the hamiltonicity of a derived graph we call the "vertex envelope" G_V^* of G. (The "vertex envelope" is defined in the next section of this paper.) Using the characterizing theorem we obtain (Theorem 8), we will show (among other results) that if G is a plane cubic graph with an independent set I of vertices such

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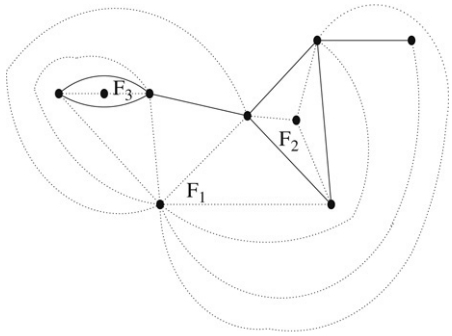

Fig. 1. $G \cup R(G)$.

that $G - I$ is a tree, then the vertex envelope of G is hamiltonian (Theorem 11). We will also show that if G is a cyclically 4-edge-connected plane cubic graph of order $n \equiv 2 \pmod 4$, then the vertex envelope of G is hamiltonian (Corollary 15).

In establishing our main results, we follow the strategy of searching for a certain subgraph in the given graph G from which to conclude hamiltonicity in the derived graph, the vertex envelope G_V^*.

We also consider related algorithmic problems. We show that, narrow though it may seem, the class of vertex envelopes of plane cubic graphs is wide enough for the question of the existence of a hamiltonian cycle to be NP-complete. The graphs used in the proof of this result have connectivity 2. This may be the root cause of the NP-completeness: We conjecture that the vertex envelope of a 3-connected plane cubic graph is always hamiltonian. We end by giving a $O(|V(X)|^2)$ algorithm for deciding whether a given cubic plane graph X is a vertex envelope.

Vertex envelopes of cubic graphs (or rather, polyhedra) are well known in physical chemistry, where they are extensively used in the study of fullerenes and tubulenes. In the literature of physical chemistry the vertex envelope of a graph G is called the "leapfrog" of G. The term – if not the construction – goes back to Fowler [11], expressing the fact that in passing from G to its envelope one "jumps" over the intermediate graph formed by G and its radial graph $R(G)$ [1]. (In polyhedral language, this is the deltahedron obtained by capping G on every face.) Applications of the leapfrog construction to toroidal polyhedra also occur in the literature of physical chemistry, e.g. Yoshida et al. [20].

2. Definitions and preliminary results

A **plane graph** is a plane embedding of a planar graph. Throughout this paper G is a loopless plane graph with multiple edges allowed.

The **boundary** $bd(F)$ **of a face** F of a plane graph G is the boundary walk of the face (in either of its orientations), allowing for repetition of vertices and/or edges when G has a cut vertex or a bridge. Generalizing the well-known construction in the 2-connected case (see [1]), we construct the **radial graph** $R(G)$ by introducing, for each face F of G, a new vertex v_F inside F and joining v_F to the vertices on the boundary of the face F in such a way that the cyclic order of the vertices on the boundary walk of F induces a cyclic order of the edges incident with v_F. Thus $G \cup R(G)$ is a triangulation of the plane. An example of this construction is shown in Fig. 1, where the edges of G and $R(G)$ are shown as solid and dashed curves, respectively.

The **vertex envelope** G_V^* of G is the dual $(G \cup R(G))^*$. In forming the dual there is a pairing of the edges of G_V^* with edges of $G \cup R(G)$; in the usual drawing of a graph and its dual, these paired edges cross. We call those edges of G_V^* which are paired with edges of G **crossing edges** of G_V^*, and we use the notation h_e for the edge of G_V^* crossing edge e of G. We say h_e **crosses** edge e and edge e **crosses** edge h_e. If the edge e separates face F_1 from a different face F_2 of G, we denote the ends of h_e as (e, F_1) and (e, F_2), with vertex (e, F_1) being the vertex drawn in face F_1 in the usual drawing of the dual. If edge e has the same face F of G on both sides, we denote the ends of h_e as (e, F) and (e, F'), the placement of the labels being arbitrary.

The edges of G_V^* which are not crossing edges form disjoint cycles. In the usual drawing of a graph and its dual, each of these cycles is contained in a face F of G and has the same edge-count as the boundary of F, bridges counted twice.

It is easily seen that the faces of G_V^* come in two varieties:

(1) Faces F_v for vertices v of G; these have $2d_G(v)$ edges on their boundaries, and these edges alternate between crossing edges and uncrossed edges. We refer to these faces as faces of **type** F_v. A face of type F_v envelopes the vertex v in the usual drawing; hence the term "vertex envelope."
(2) Faces F' corresponding to faces F of G; for these, $|E(bd(F))| = |E(bd(F'))|$. None of the boundary edges of these faces are crossed by edges of G. We refer to these faces as faces **derived from** faces of G.

We see that G_V^* is bipartite if and only if G is bipartite. Also, a crossing edge of a bridge of G is a chord of the cycle of G_V^* derived from the face having the bridge on its boundary.

Proposition 1. *G_V^* is a planar cubic graph of order $2|E(G)|$ and size $3|E(G)|$.*

Proof. This follows immediately from the construction of G_V^*. $\square$