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On wreathed lexicographic products of graphs☆

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ABSTRACT

This paper proves a necessary and sufficient condition for the endomorphism monoid *End* $G[H]$ of a lexicographic product $G[H]$ of graphs G, H to be the wreath product *End* $G \wr$ *End* H of the monoids *End* G and *End* H. The paper also gives respective necessary and sufficient conditions for specialized cases such as for unretractive or triangle-free graphs G.

1. Introduction

Harary introduced the lexicographic product of graphs in [9]. His work was inspired by the success in many combinatorial applications of the wreath product of groups that was pioneered by Polya [31]. Harary based the definition of the lexicographic product on earlier work of Frucht [6] and Zykov [35]. These authors had used special cases of the lexicographic product, namely $\overline{K_n}[H]$ and $K_2[H]$, respectively. In these cases, the automorphism group of the product graph turned out to be a wreath product that involved the automorphism group of the graph H. Harary aimed at defining a product of two graphs, namely the lexicographic product, whose automorphism group is equal to the wreath product of the automorphism groups of its factors. As it turned out, this equation does not hold for certain graphs. A first related result was given by Sabidussi in [32] and then generalized to a larger class of pairs of graphs by Hemminger in [13] and [14]. In [15], Hemminger also extended that work so that it covered the lexicographic product of a graph with a family of graphs, also known as a join.

Studying wreath products, according to [17, p. 5], in special cases goes back to the 19th century to the work of Cauchy, Jordan and Netto. In the 1930s, according to that source, Polya and Sperber (among others) came to consider wreath products of permutation groups. According to Kilp et al. [23, p. 165], the earliest representative of that group of publications is Loewy's paper [29]. A brief discussion of wreath products of monoids is provided below. For more details, see [23,20].

Hell introduced the category of graphs in [11]. After that graph morphisms were studied, for example, in [8] and [12]. Notable contributions were also made by Knauer and various of his co-authors or colleagues in [24,28,25] (see also [19, 30,34]). Knauer et al. in particular introduced different kinds of graph morphisms and attempted to characterize graphs by their endospectra, i.e., six-tuples of constraints of endomorphism sets ranging from automorphisms to endomorphisms [1,26,27]. The results proved in this paper may be helpful for a better understanding of endomorphism monoids of graphs or the construction of graphs whose endomorphism monoid has certain prescribed algebraic properties such as being a group or von Neumann regular.

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2. Preliminaries

We denote the power set of a set S by $\mathcal{P}(S)$. For sets S and T we denote the set of mappings from S to T by $\mathcal{F}(S,T)$. The image of $f \in \mathcal{F}(S,T)$ and its restriction to $U \subseteq S$ are denoted $\mathrm{im}(f)$ and $f|_U$, respectively. The cardinality of S and the identity mapping on S are denoted by $|S|$ and 1_S, respectively.

Lemma 1 (*[33]*). *For any element s of a finite semigroup S there exists a positive integer n such that s^n is idempotent, i.e., $s^n s^n = s^n$.*

In this paper only finite, undirected, and simple graphs are considered. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The edge $e \in E(G)$ incident with the vertices g and g' is denoted by gg'. Many standard concepts and notations will be used below, roughly as in [7, Ch. 1]. The set of neighbors of a vertex g is denoted by $\mathcal{N}(g)$. A vertex g of graph G is called *common neighbor* of a set S of vertices of G if $g \in \cap_{s\in S} \mathcal{N}(s)$. For each integer n, the symbols K_n, D_n, P_n and C_n stand for the *complete*, the *discrete* graph, the *path*, and the *cycle* of order n, respectively. The K_3 is also called a *triangle*. The *complement* of G is denoted by $\overline{G}$. For any subset S of $V(G)$ the *induced subgraph* $\langle S\rangle_G$ has the vertex set S and the edge set $\{hh' \mid h, h' \in S, hh' \in E(G)\}$.

Let G and H be graphs. A *graph morphism* $r : G \to H$ is an edge preserving vertex mapping. The set of graph morphisms from G into H is denoted by $Hom(G,H)$, $Hom(G,G)$ is denoted by $End\ G$, and each element thereof is called an *endomorphism*. Rather than $1_{V(G)}$, we usually write 1_G. A graph H is called a *retract* of a graph G if there are morphisms $\iota : H \to G$ and $\pi : G \to H$ such that $\pi \circ \iota = 1_H$. In this case π and ι are called a *retraction* and a *co-retraction*, respectively. The graph G is also called a *co-retract* of H, and H is called a *retract* of G. A morphism which is a retraction and a co-retraction is called an *isomorphism*.

The *chromatic number* $\chi(G)$ is known to be equal to $\min\{n \mid Hom(G,K_n) \neq \emptyset\}$. A graph G is known to be bipartite if and only if it contains no odd cycles (see, for example, [10, Theorem 2.4]). For a bipartite graph G, obviously $\chi(G) \in \{1,2\}$. Corollary 8 of [22] states that $\chi(G[H]) \geq \chi(G) + 2\chi(H) - 2$, for a graph G with $E(G) \neq \emptyset$ and a graph H. Therefore, for an odd cycle $\mathcal{O}$ and a bipartite graph $\mathcal{B}$ with $E(\mathcal{B}) \neq \emptyset$ with $\chi(\mathcal{O}) = 3$ and $\chi(\mathcal{B}) = 2$, we have $\chi(\mathcal{O}[H]) > 2\chi(H)$ and $\chi(\mathcal{B}[H]) = 2\chi(H)$. Obviously, $\chi(H) \geq \chi(G)$ if there exists a morphism $\phi : G \to H$. A morphism $\phi : G \to H$ is called *locally strong* if, for all $\phi(g)\phi(g') \in E(H)$ and for all $\gamma \in \phi^{-1}(\phi(g))$, there exists a $\gamma' \in \phi^{-1}(\phi(g'))$ such that $\gamma\gamma' \in E(G)$. A morphism $\phi : G \to H$ is called *strong* if, for any $g, g' \in V(G)$ with $\phi(g)\phi(g') \in E(H)$, then $gg' \in E(G)$. Obviously, strong morphisms are locally strong. These concepts were introduced by Böttcher and Knauer in [1].

An isomorphism $\phi : G \to G$ is called an *automorphism* and the automorphism group of G is denoted as $Aut\ G$. Obviously, $1_G \in Aut\ G \subseteq End\ G$. It is well known and easy to show that the automorphisms of a graph are exactly its bijective endomorphisms. A graph G is called *unretractive* or a *core graph* if $End\ G = Aut\ G$. In [8] and in [18, Theorem II.1.5], it has been shown that any two unretractive retracts of a given graph are isomorphic to each other. Each instance of this isomorphism class of a graph G is denoted by C_G and called the *core* of G. This term is used in [8] and goes back to Hell and Nešetřil. Necessary and sufficient conditions for unretractive graphs are given in [20,21]. Because of Lemma 1, each endomorphism $\phi \in End\ G$ has an idempotent power ϕ^n. Therefore a graph G is unretractive if and only if 1_G is its only idempotent endomorphism. A graph G is called *rigid* if $End\ G = \{1_G\}$. Obviously, rigid graphs are unretractive.

It is also well known that the retracts of a graph G up to isomorphism are exactly the induced subgraphs $\langle \mathrm{im}(\phi)\rangle_G$ of homomorphic images of idempotent endomorphisms of G; see, for example, [18, Remark II.1.2]. The following relations were defined by Sabidussi for any graph G (see [32]): $R_G = \{(g,g') \in V(G) \times V(G) \mid \mathcal{N}_G(g) = \mathcal{N}_G(g')\}$, $S_G = R_{\overline{G}}$. It was shown by Sabidussi that $(g,g') \in S_G$ if and only if $\mathcal{N}_G(g) \cup \{g\} = \mathcal{N}_G(g') \cup \{g'\}$. A subset $S \subseteq V(G)$ of the vertex set $V(G)$ of a graph G is called a *multi-cone* or externally related (the latter term is used by Imrich) if $\mathcal{N}_G(s) \setminus S = \mathcal{N}_G(s') \setminus S$, $\forall s, s' \in S$. It is easy to see that the classes of R_G and S_G are multi-cones and are independent or complete respectively. The *identity relation* on a set S will be denoted by Δ_S (or Δ if there is no danger of confusion).

Let G be a graph and $\mathcal{H} = \{H_g\}_{g\in V(G)}$ be a family of graphs. The graph L with $V(L) = \{(g,h) \in V(G) \times \cup_{g\in V(G)} V(H_g) \mid h \in V(H_g)\}$ and $E(L) = \{\{(g,h),(g',h')\} \mid g = g'$ and $hh' \in E(H_g)$, or $gg' \in E(G)\}$ is called the *lexicographic product* of G with the family $\mathcal{H}$. It usually is called the *join* of G and $\mathcal{H}$, and is denoted by $G[\mathcal{H}]$ or $G[H_g \mid g \in V(G)]$. If there is a graph H such that $H_g = H$, for all $g \in V(G)$, then $G[\mathcal{H}]$ is denoted by $G[H]$. Rather than $(\{g\},\emptyset)[H]$, we write $g[H]$. The lexicographic product $K_2[H_1,H_2]$ is often denoted by $H_1 + H_2$ and called the *sum* of H_1 and H_2. Let G and M be graphs and $\{H_g\}_{g\in V(G)}$, $\{N_m\}_{m\in V(M)}$ be families of graphs. Let $L = G[H_g \mid g \in V(G)]$, $Q = M[N_m \mid m \in V(M)]$ and let $\phi : L \to Q$ be a morphism. The *spectral mapping* $\Sigma(\phi)$ of ϕ is the mapping $\Sigma(\phi) : V(G) \to \mathcal{P}(V(M))$, $g \mapsto \{m \in V(M) \mid \exists h \in V(H_g), n \in N_m$ with $\phi(g,h) = (m,n)\}$. The *ϕ-spectrum* of graph H_g is the set $\Sigma(\phi)(g)$. Following Hemminger (see [15]), we call the morphism ϕ *natural* if $|\Sigma(\phi)(g)| = 1$, for all $g \in V(G)$. We call ϕ *full* if $m[N_m] \subseteq \phi(g[H_g])$ or $m[N_m] \cap \phi(g[H_g]) = \emptyset$, for all $g \in V(G)$ and $m \in V(M)$. The set of strong or full morphisms from L to Q is denoted by $Strong(L,Q)$ or $Full(L,Q)$, respectively.

Let M be a monoid with the identity element 1_M, S a set, and $ms \in S$ for all $m \in M$, $s \in S$. Then S is called an *M-act* if $1_M s = s$ and $(mm')s = m(m's)$, for all $s \in S$, $m, m' \in M$. Let M, N be monoids and S, T be an M-act and N-act respectively. It is well known that $W = M \times \mathcal{F}(S,N)$ with the multiplication $(m,f)(n,g) := (mn, f_n g)$, with $f_n g : S \to N$, $s \mapsto f(ns)g(s)$ is a monoid. The identity element of this monoid is $(1_M, c_1)$, where $c_1(s) = 1_N$, for all $s \in S$. W is called the *wreath product* of M with N over S. It is denoted by $(M \wr N \mid S)$ or, if there is no danger of confusion, by $(M \wr N)$. Provided the respective group inverse elements exist in M and N, the inverse of (m,f) is $(m,f)^{-1} = (m^{-1}, g)$, with $g(s) = f(m^{-1}s)^{-1}$, for all