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G-parking functions, acyclic orientations and spanning trees

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ABSTRACT

Given an undirected graph $G = (V, E)$, and a designated vertex $q \in V$, the notion of a G-parking function (with respect to q) was independently developed and studied by various authors, and has recently gained renewed attention. This notion generalizes the classical notion of a parking function associated with the complete graph. In this work, we study the properties of *maximum* G-parking functions and provide a new bijection between them and the set of spanning trees of G with no broken circuit. As a case study, we specialize some of our results to the graph corresponding to the discrete n-cube Q_n. We present the article in an expository self-contained form, since we found the combinatorial aspects of G-parking functions somewhat scattered in the literature, typically treated in conjunction with sandpile models and closely related chip-firing games.

1. Introduction

The classical parking functions provide a bijective correspondence between the spanning trees of the complete graph K_n and certain integer-valued functions on the vertices of K_n. A notion of parking functions corresponding to the spanning trees of an arbitrary graph G is more recent and has been independently developed in physics and combinatorics. It was introduced by Bak, Tang and Wisenfeld [3] as a self-organized sandpile model on grids, and was generalized to arbitrary graphs by Dhar [17]. See Definition 2.1 below for the precise definition of a G-parking function, associated with a connected graph G.

This notion is already rather powerful; besides generalizing the classical parking function from K_n to an arbitrary graph, it has been investigated in the context of chip-firing games [8,29,30] and the Tutte polynomial [9,14] in discrete mathematics, and also investigated in algebra and related fields [2,15,16,20,31]. However, some of the combinatorial aspects of this topic appear somewhat scattered in the literature.

Several fundamental results concerning the recurrent configurations of chip-firing can be derived without the chip-firing context and terminology. For this reason, we shy away from introducing and discussing the chip-firing terminology. Instead, in this article we describe various interpretations of the G-parking functions in the most elementary combinatorial ways. Using a natural partial order $\prec$ on the set $\mathcal{P}(G, q)$ of parking functions, we consider the maximal elements in this poset $\left(\mathcal{P}(G, q), \prec\right)$. Much of our focus in this paper is on understanding the properties of such maximal parking functions. The first result we describe (see Theorem 4.1) provides a new bijection between the maximal parking functions in the poset and the set $\mathcal{A}(G; q)$ of acyclic orientations of G with a unique source at q. En route, we describe what we call an Extended Dhar algorithm (since it is an extension of an algorithm due to Dhar [17] to recognize G-parking functions) in providing an acyclic orientation corresponding to a maximal parking function. We review various combinatorial consequences and

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algebraic connections of this correspondence. For example, using known results (namely those of Greene and Zaslavsky [24] and more recent work of Gebhard and Sagan [21]), we further identify a 1–1 correspondence between the set of maximal parking functions and the set of spanning trees with no "broken circuits," or equivalently, the set of "safe" spanning trees ; see Section 4 for the definitions of these terms. In this paper, we provide a much simpler bijection (compared to [21]) between the set of safe trees and the set of acyclic orientations with a unique sink (or equivalently, a unique source). Furthermore we generalize this bijection to one between all spanning trees and all G-parking functions which *preserves* the bijection between safe trees and maximal G-parking functions. We must remark here that other bijective proofs between the set of G-parking functions and the set of spanning trees of G (for arbitrary connected G) have been given by Chebikin and Pylyavskyy [13]. However, to our knowledge, the simpler bijection we report here, in Theorem 4.2 below, and its generalization given in Theorem 4.6, are indeed new.

As an additional contribution, we describe a simple way to generate maximal parking functions in the Cartesian product graph $G_1 \square G_2$, using maximal functions in the (factor) graphs G_1 and G_2. We then specialize our study to understanding the parking functions in the discrete n-cube Q_n on 2^n vertices. By describing certain special constructions of maximal parking functions f on Q_n, we obtain a natural description of a set, dom(f), of parking functions — those dominated, in the partial order given by $\prec$, by a special maximal parking function f. Interestingly enough we shall deduce (see Theorem 5.2) that

$$|\mathrm{dom}(f)| = \prod_{k=2}^{n} k^{\binom{n}{k}}, \tag{1.1}$$

while it is a well-known fact that

$$|\mathcal{P}(Q_n, q)| = \prod_{k=2}^{n} (2k)^{\binom{n}{k}} = 2^{2^n-n-1} \prod_{k=2}^{n} k^{\binom{n}{k}}. \tag{1.2}$$

Recall that (1.2) corresponds to the total number of spanning trees of Q_n (see Eq. 5.85 in [32]), using the matrix-tree theorem and the explicit knowledge of the corresponding eigenvalues, to help evaluate the determinantal formula. In light of the fact that finding a bijective proof accounting for the number of spanning trees of Q_n has been open for several years, we hope this is a nontrivial step towards such a proof.

The paper is organized as follows. In Section 2, we review some preliminaries, including Dhar's burn criterion, which determines whether a given function is a parking function. In Section 3, we show the bijection between maximum parking functions and acyclic orientations with a unique source. In Section 4.1, we describe our new and simpler bijection between the set of acyclic orientations with a unique sink and the set of safe trees. In Section 5.1, we describe a construction of maximum parking functions on Cartesian products of graphs. In Section 5.2, we focus our study on the n-cube Q_n, and provide some explicit constructions of maximum parking functions and related bounds. In Section 6, for expository purposes we review a bijection between *diffuse states* (introduced in the context of chip-firing) and acyclic orientations of a graph. We conclude with some remarks on research in future directions and a few open problems in Section 7.

2. *G*-Parking functions and Dhar's burn criteria

In this section we recall the definition of a G-parking function and review Dhar's (burning) algorithm that can be used to determine whether an integer-valued function on the vertices of G is a G-parking function.

Definition 2.1. For a connected graph G, a G-parking function relative to vertex $q \in G$ is a function $f : V(G) \to \mathbb{Z}_{\geq -1}$ such that $f(q) = -1$ and for every non-empty $A \subseteq V(G) \setminus \{q\}$, there exists $v \in A$ such that $0 \leq f(v) < d_{\overline{A}}(v)$, where $d_{\overline{A}}(v)$ is the number of edges $e = vw$ with $w \notin A$.

Remark 2.1. Herein, we have modified the standard definition of a G-parking function somewhat. The function f is now defined on all of $V(G)$ instead of restricted to simply $V(G) \setminus \{q\}$ in order to improve the compatibility between G-parking functions and Cartesian product graphs such as Q_n; due to this change, for f to be a G-parking function, $f(q) = -1$ necessarily.

Proposition 2.1. *If for a function $f : V(G) \setminus \{q\} \to \mathbb{Z}_{\geq 0}$, for every non-empty connected subgraph $A \subseteq G \setminus \{q\}$, there exists $v \in V(A)$ such that $f(v) < d_{\overline{A}}(v)$, then f is a G-parking function.*

Proof. Assume that, for all connected $A \subseteq G \setminus \{q\}$, that there exists $v \in V(A)$ such that $f(v) < d_{\overline{A}}(v)$. Proceeding by contradiction, suppose that there is some disconnected $B \subseteq G \setminus \{q\}$ such that $f(v) \geq d_{\overline{B}}(v)$ for every $v \in V(B)$. Consider then any connected component C of B. Since C is connected we have, by the hypothesis of the proposition, that $f(v) < d_{\overline{C}}(v)$, for some vertex v in C. Thus $d_{\overline{B}}(v) < d_{\overline{C}}(v)$, implying that there is a vertex u in $\overline{C} \setminus \overline{B}$ such that v and u are connected by an edge in G; otherwise, either $f(v) \geq d_{\overline{C}}(v)$ or $f(v) < d_{\overline{B}}(v)$. This contradicts the choice of C. ■

Throughout we assume that the reference vertex q is fixed, and we always consider parking functions with respect to this fixed vertex q, without necessarily bringing an explicit reference to it.

A natural question to ask is whether a given integer-valued function on the vertices of G can easily be tested for being a G-parking function. In the context of the so-called sandpile models, Dhar [17] provided an algorithm, which can be