Discrete Mathematics 312 (2012) 1353–1366

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Closed trail decompositions of some classes of regular graphs

P. Paulraja *, S. Sampath Kumar

Department of Mathematics, Annamalai University, Annamalainagar - 608 002, India

ARTICLE INFO

Article history:
Received 1 July 2011
Received in revised form 13 December 2011
Accepted 16 December 2011
Available online 20 January 2012

Keywords:
Closed trail decomposition
Tensor product
Wreath product
Hamilton cycle
Intersection graph

ABSTRACT

If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$, then we say that $H_1, H_2, \ldots, H_k$ *decompose* G. If each $H_i \cong H$, then we say that H decomposes G and we denote it by $H|G$. If each H_i is a closed trail, then the decomposition is called a *closed trail decomposition* of G. In this paper, we consider the decomposition of a complete equipartite graph with multiplicity λ, that is, $(K_m \circ \overline{K}_n)(\lambda)$, into closed trails of lengths $pm_1, pm_2, \ldots, pm_k$, where p is an odd prime number or $p = 4$, $\sum_{i=1}^{k} pm_i$ is equal to the number of edges of the graph and $\circ$ denotes the wreath product of graphs. A similar result is also proved for $(K_m \times K_n)(\lambda)$, where $\times$ denotes the tensor product of graphs, if there exists a p-cycle decomposition of the graph. We obtain the following corollary: if $k \geq 3$ divides the number of edges of the even regular graph $(K_m \circ \overline{K}_n)(\lambda)$, then it has a T_k-decomposition, where T_k denotes a closed trail of length k. For $m, n \geq 3$, this corollary subsumes the main results of the papers [A. Burgess, M. Šajna, Closed trail decompositions of complete equipartite graphs, J. Combin. Des. 17 (2009) 374–403]; [B.R. Smith, Decomposing complete equipartite graphs into closed trails of length k, Graphs Combin. 26 (2010) 133–140]. We have also partially obtained some results on T_k-decomposition of $(K_m \times K_n)(\lambda)$.

1. Introduction

All graphs considered here are simple and finite unless otherwise stated. Let C_k (resp.T_k) denote a cycle (resp. closed trail) of length k. Let P_k denote a path on k vertices. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$, then we say that $H_1, H_2, \ldots, H_k$ *decompose* G. If each $H_i \cong H$, then we say that H decomposes G and we denote it by $H \mid G$. If the edge set of G can be partitioned into cycles of length k, then we write $C_k \mid G$, and in this case we say that G has a *C_k-decomposition* or a *k-cycle decomposition*. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. A *k-factor* of G is a spanning subgraph H of G such that each component of H is a k-regular subgraph of G. In what follows, a *C_k-factor* is a 2-factor in which each component is a C_k. A partition of the edge set of G into C_k-factors is called a *C_k-factorization* of G and we denote it by $C_k \| G$. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S \rangle$. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$, then we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

For two graphs G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Similarly, the *wreath product* of the graphs G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G, or $g_1 = g_2$ and h_1h_2 is an edge in H. Let G and H be simple graphs with vertex sets $V(G) = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{n-1}\}$. Then $V(G \times H) = V(G) \times V(H)$ and for our convenience, we shall denote the vertices of $G \times H$ by $\{x_{i,j} \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$,

* Corresponding author.
E-mail addresses: pprajaau@sify.com (P. Paulraja), ssamkumar.2008@gmail.com (S. Sampath Kumar).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.12.015

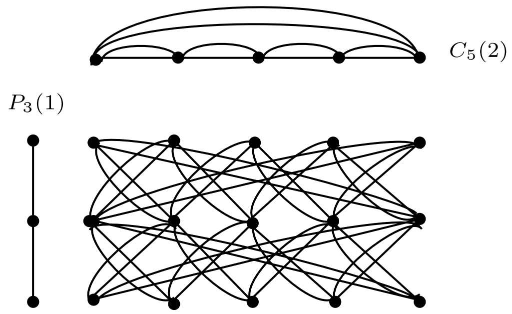

Fig. 1. The graph $P_3(1) \times C_5(2) \cong (P_3 \times C_5)(2)$.

where $x_{i,j}$ stands for the vertex (x_i, y_j). Similarly, we denote the vertices of $V(G \circ H)$ also. It is well known that the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. Clearly, $K_m \times K_n$ can be obtained from $K_m \circ \overline{K}_n$ by deleting edges of n vertex disjoint copies of K_m. For an integer $n \geq 2$, by the notation nG, we mean n edge disjoint isomorphic copies of G (where the vertices of the copies of G may have intersection or not). A *multigraph* $G(\lambda)$ is the graph obtained from G by replacing each edge of G by λ parallel edges.

For two loopless multigraphs $G(\lambda)$ and $H(\mu)$, the tensor product, denoted by $G(\lambda) \times H(\mu)$, has the vertex set $V(G) \times V(H)$ and its edge set is described as follows: if $e = g_1g_2$ is an edge of multiplicity λ in $G(\lambda)$ and $f = h_1h_2$ is an edge of multiplicity μ in $H(\mu)$, then corresponding to these edges there are edges $(g_1, h_1)(g_2, h_2)$ and $(g_1, h_2)(g_2, h_1)$ each of multiplicity $\lambda\mu$ in $G(\lambda) \times H(\mu)$ and $G(\lambda) \times H(\mu)$ is isomorphic to $(G \times H)(\lambda\mu)$, see Fig. 1. Hence $G(\lambda) \times H \cong G \times H(\lambda) \cong (G \times H)(\lambda)$. Similarly, we have $G(\lambda) \circ \overline{K}_n \cong (G \circ \overline{K}_n)(\lambda)$.

For disjoint subsets $A, B \subset V(G)$, $E(A, B)$ denotes the set of edges of G having one end in A and the other end in B. Let $K_{\ell(a),b}$ denote the complete $(\ell + 1)$-partite graph with ℓ partite sets of size a each and, one partite set of size b. Definitions which are not given here can be found in [3] or [12].

A connected even regular graph G is said to be *Arbitrarily Decomposable into Closed Trails*, or *ADCT* for short, if given any (multi) set $\{m_1, m_2, \ldots, m_k\}$ of positive integers greater than 2, with the property that G contains a closed trail of length m_i, $1 \leq i \leq m$, and also satisfying $\sum_{i=1}^{k} m_i = |E(G)|$, then the graph G has an edge-disjoint decomposition into closed trails of lengths $m_1, m_2, \ldots, m_k$.

Decomposition of a graph into closed trails is not new. In [4], Balister showed that K_n, n odd and, $K_n - F$, where F is a 1-factor of K_n, when n is even, are both *ADCT*. By *ADDCT* we mean *Arbitrarily Decomposable into Directed Closed Trails*. Further, in [5], Balister proved that K_n^*, the complete symmetric digraph on n vertices, is *ADDCT*. Billington and Cavenagh showed that the complete tripartite graph $K_3 \circ \overline{K}_n$ is ADCT; see [8]. In [6], decompositions of complete multipartite graphs into cycles or closed trails are dealt with in detail. Recently, Burgess and Šajna [14], and independently Smith [24], proved that for $m, k \geq 3$, $T_k \mid K_m \circ \overline{K}_n$ whenever k divides the number of edges of $K_m \circ \overline{K}_n$ and $(m - 1)n$ is even, where T_k is a closed trail of length k.

Here, we consider the problem of decomposing $(K_m \times K_n)(\lambda)$, $m \geq 3$, $n \geq 3$ and $(K_m \circ \overline{K}_n)(\lambda)$, $m \geq 3$, $n \geq 3$ into closed trails. We prove the following theorems.

Theorem 1.1. *Let $m, n \geq 3$ and let $p \geq 3$ be a prime number or $p = 4$. If there exists a p-cycle decomposition of $(K_m \times K_n)(\lambda)$, then it has a decomposition into closed trails of lengths $pm_1, pm_2, \ldots, pm_k$, where $m_i \geq 1$ and $\sum_{i=1}^{k} pm_i = \frac{\lambda mn(m-1)(n-1)}{2}$.*

Corollary 1.1. *Let $m, n, k \geq 3$ and for $k \neq 2^\alpha$, let $k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, be the prime factorization of k. Let k divide the number of edges of $(K_m \times K_n)(\lambda)$ and $\lambda(m-1)(n-1)$ is even. If there exists a p_i-cycle decomposition of $(K_m \times K_n)(\lambda)$, for some i, then $T_k \mid (K_m \times K_n)(\lambda)$, where T_k denotes a closed trail of length k. If $k = 2^\alpha$ and if $C_4 \mid (K_m \times K_n)(\lambda)$, then $T_k \mid (K_m \times K_n)(\lambda)$.*

Corollary 1.2. *If $m, n, k \geq 3$ so that $(m-1)(n-1)$ is even and $k \mid \frac{mn(m-1)(n-1)}{2}$, then $K_m \times K_n$ has a decomposition into closed trails of length k.*

Theorem 1.2. *Let $m, n \geq 3$, and let $p \geq 3$ be a prime number or $p = 4$. If $(K_m \circ \overline{K}_n)(\lambda)$ is an even regular graph and if $p \mid \lambda \binom{m}{2} n^2$, then it has a decomposition into closed trails of lengths $pm_1, pm_2, \ldots, pm_k$, where $m_i \geq 1$ and $\sum_{i=1}^{k} pm_i = \lambda \binom{m}{2} n^2$.*

Corollary 1.3. *If $m, n, k \geq 3$ so that $\lambda(m-1)n$ is even and $k \mid \lambda \binom{m}{2} n^2$, then $(K_m \circ \overline{K}_n)(\lambda)$ has a decomposition into closed trails of length k.*

The main results of [14,24] can be deduced as a corollary by substituting $\lambda = 1$ in Corollary 1.3.

Corollary 1.4 ([14,24]). *If $m, n, k \geq 3$ so that $(m-1)n$ is even and $k \mid \binom{m}{2} n^2$, then $K_m \circ \overline{K}_n$ has a decomposition into closed trails of length k.*