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A multiset hook length formula and some applications

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ABSTRACT

A multiset hook length formula for integer partitions is established by using combinatorial manipulation. As special cases, we rederive three hook length formulas, two of them obtained by Nekrasov–Okounkov, the third one by Iqbal, Nazir, Raza and Saleem, who have made use of the cyclic symmetry of the topological vertex. A multiset hook-content formula is also proved.

1. Introduction

Recently, an elementary proof of the Nekrasov–Okounkov hook length formula [18] was given by the second author in [8], using the Macdonald identities for A_t (see [15]). A crucial step of that proof is the construction of a bijection between t-cores and integer vectors satisfying some additional properties. Several further papers related to the Nekrasov–Okounkov formula have been published. See, *e.g.*, [24,3,5,4,10,23,20,11].

In the present paper, we again take up the study of the Nekrasov–Okounkov formula and obtain several results in the following directions. (1) The bijection between t-cores and integer vectors is constructed for any positive integer t, while in [8], t had to be an odd positive integer. (2) That bijection is shown to satisfy a multiset hook length formula (Theorem 1) with a functional parameter τ by using a geometric model, called "exploded tableau". The result in [8] corresponds to the special case $\tau(x) = x$. (3) A multiset hook length formula provides another special case when taking $\tau = \sin$, namely Theorem 2. (4) Three hook length formulas are derived (Corollaries 7 and 8, Theorem 5), the first two previously obtained by Nekrasov–Okounkov [18], the third one by Iqbal et al. [11]. (5) Theorem 2 provides a unified formula for the Nekrasov–Okounkov formula and the classical Jacobi triple product identity [2, p. 21], [12, p. 20]. This formula solves Problem 6.4 in [7]. (6) A multiset hook-content hook length formula is also given in Section 6.

The basic notions needed here can be found in [16, p. 1], [22, p. 287], [13, p. 1], [12, p. 59], and [2, p. 1]. A *partition* λ of *size* n and of *length* ℓ is a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. We write $n = |\lambda|$, $\ell(\lambda) = \ell$ and $\lambda_i = 0$ for $i \geq \ell + 1$. The set of all partitions of size n is denoted by $\mathcal{P}(n)$. The set of all partitions is denoted by $\mathcal{P}$, so that $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}(n)$. The *hook length multiset* of λ, denoted by $\mathcal{H}(\lambda)$, is the multiset of all hook lengths of λ. Let t be a positive integer. We write $\mathcal{H}_t(\lambda) = \{h \mid h \in \mathcal{H}(\lambda), h \equiv 0 (\mathrm{mod}\ t)\}$. A partition λ is a *t-core* if $\mathcal{H}_t(\lambda) = \emptyset$ (see [12, p.69, p.612], [22, p. 468]). For example, $\lambda = (6, 3, 3, 2)$ is a partition of size 14 and of length 4. We have $\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$ and $\mathcal{H}_2(\lambda) = \{2, 4, 4, 2, 8, 6, 2\}$ (see also [8]).

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Table 1
The example $\lambda = (8, 4, 3, 2, 2, 1)$ with $t = 5$. Note that this also gives $W_1(\lambda)$, $V_1(\lambda)$, etc. since $W(\lambda) = W_1(\lambda)$, etc.

λ	$(8, 4, 3, 2, 2, 1)$
$W(\lambda)$	$\{10, 5, 3, 1, 0, -2, -4, -5, -6, -7, -8, -9, -10, -11, \dots\}$
$V(\lambda)$	$\{10, 3, 1, -6, -8\}$
$W^\dagger(\lambda)$	$\{5, 0, -2, -4, -5, -7, -9, -10, -11, \dots\}$
$M(\lambda)$	10
$m(\lambda)$	-4
$C(\lambda)$	$\{9, 8, 7, 6, 4, 2, -1, -3\}$
λ^*	$(6, 5, 3, 2, 1, 1, 1, 1)$
$W_2(\lambda)$	$\{8, 6, 3, 1, -1, -2, -3, -4, -6, -7, -8, -9, -10, -11, -12, \dots\}$
$V_2(\lambda)$	$\{8, 6, -1, -3, -10\}$
$W_2^\dagger(\lambda)$	$\{3, 1, -2, -4, -6, -7, -8, -9, -11, -12, \dots\}$
$M_2(\lambda)$	8
$m_2(\lambda)$	-6
$C_2(\lambda)$	$\{7, 5, 4, 2, 0, -5\}$

Let t be a positive integer and $t_0 = 0$ (resp. $t_0 = 1/2$) if t is odd (resp. even). Consider the set of (half-)integers $\mathbb{Z}' = t_0 + \mathbb{Z}$. Each vector of (half-)integers $\vec{V} = (v_0, v_1, \dots, v_{t-1}) \in \mathbb{Z}'^t$ is called a V_t-*coding* if the following conditions hold: (i) $\{v_i - i \bmod t : i = 0, \dots, t-1\}$ is equal to $t_0 + \{0, 1, \dots, t-1\}$, (ii) $v_0 + v_1 + \cdots + v_{t-1} = 0$, (iii) $v_0 > v_1 > \cdots > v_{t-1}$.

Theorem 1. *Let t be a positive integer and $\tau : \mathbb{Z} \to F$ be any weight function from $\mathbb{Z}$ to a field F. Then, there is a bijection $\phi_t : \lambda \mapsto \vec{V} = (v_0, v_1, \dots, v_{t-1})$ from t-cores onto V_t-codings such that*

$$|\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \cdots + v_{t-1}^2) - \frac{t^2-1}{24} \tag{1}$$

and

$$\prod_{h \in \mathcal{H}(\lambda)} \frac{\tau(h-t)\tau(h+t)}{\tau(h)^2} = \prod_{i=1}^{t-1} \frac{\tau(-i)^{\beta_i(\lambda)}}{\tau(i)^{\beta_i(\lambda)+t-i}} \prod_{0 \le i < j \le t-1} \tau(v_i - v_j), \tag{2}$$

where $\beta_i(\lambda) = \#\{\square \in \lambda : h(\square) = t - i\}$.

The proof of Theorem 1 is given in Section 3. With the weight function $\tau = \sin$, an odd function, we get the specialization stated in the next theorem. Its proof is given in Section 5.

Theorem 2. *For any positive integer r and any complex numbers z, t, we have*

$$\sum_{\lambda} q^{|\lambda|} \prod_{h \in \mathcal{H}_r(\lambda)} \left(1 - \frac{\sin^2(tz)}{\sin^2(hz)}\right) = \exp \sum_{k=1}^{\infty} \left(\frac{q^k}{k(1-q^k)} - \frac{rq^{rk}}{k(1-q^{rk})} \frac{\sin^2(tkz)}{\sin^2(rkz)} \right). \tag{3}$$

Some specializations of Eq. (3) are given in Section 4.

2. Exploded tableau

With each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ and each positive integer t we associate several sets of (half-)integers. All these concepts will be illustrated for the case $\lambda = (8, 4, 3, 2, 2, 1)$ and $t = 5$ (see Table 1). Note that this case is special, as λ itself is a t-core, but this property will be assumed most of the time.

The W-*set* of λ is a translation of the shifted parts, defined to be the set of all integers of the form $\lambda_i - i + (t+1)/2$ for $i \in \mathbb{N} \setminus 0$ (the partition λ is viewed as an infinite non-increasing sequence trailing with zeros). We denote this set by $W(\lambda)$. It is immediate that $W(\lambda) \subset \mathbb{Z}'$. It is also clear that there exists a smallest (half-)integral $M = M(\lambda)$ and a largest (half-)integral $m = m(\lambda)$ such that $\{m, m-1, \dots\} \subseteq W(\lambda) \subseteq \{M, M-1, \dots\}$.

We say that an element x in a set X is t-*maximal* if it is the largest in its congruence class modulo t. If t is even, we have $W(\lambda) \subset \frac{\mathbb{Z}}{2}$. By "congruence classes mod t", we then mean the congruence classes mod t of $1/2, 3/2, \dots, t - 1/2$. The set of t-maximal elements is denoted by $t\text{-}\max(X)$. In the cases further considered, congruence classes will always contain an element, so no maximum will ever be taken over an empty set. It is then clear that $|t\text{-}\max(X)| = t$.

We define the V-*set* $V(\lambda)$ of λ by $V(\lambda) := t\text{-}\max(W(\lambda))$. It is easily seen from the definition of $m(\lambda)$ that no congruence class modulo t can be empty. We also set $W^\dagger(\lambda) = W(\lambda) \setminus V(\lambda)$. If $V(\lambda)$ is sorted by decreasing order, we get a V_t-coding (as proved in Eq. (8)), that will be denoted by $\vec{V}(\lambda) = \phi_t(\lambda)$. Thus, the bijection ϕ_t required in Theorem 1 is constructed.

We also define the complementary set $C(\lambda) := \{M, M-1, \dots\} \setminus W(\lambda)$, so that the disjoint union $W^\dagger(\lambda) \cup V(\lambda) \cup C(\lambda)$ is equal to $\{M, M-1, \dots\}$. Note that $m(\lambda) = \min C(\lambda) - 1$.

The invariants previously defined, such as $V(\lambda), W(\lambda), \dots$ will also be given the subscript "1", as in $V_1(\lambda), W_1(\lambda), \dots$. The invariants attached to the conjugate partition λ^*, such as $V(\lambda^*), W(\lambda^*), \dots$ will then be written $V_2(\lambda), W_2(\lambda), \dots$.