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2-factors and independent sets on claw-free graphs

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ABSTRACT

In this paper, we show that if G is an l-connected claw-free graph with minimum degree at least three and $l \in \{2, 3\}$, then for any maximum independent set S, there exists a 2-factor in which each cycle contains at least $l-1$ vertices in S.

1. Introduction

In this paper, we consider finite graphs. If no ambiguity can arise, we denote simply the order $|G|$ of G by n, the minimum degree $\delta(G)$ by δ and the independence number $\alpha(G)$ by α. All notation and terminology not explained in this paper is given in [1,4].

A 2-*factor* of a graph G is a spanning 2-regular subgraph of G. Choudum and Paulraj [3] and Egawa and Ota [5] independently showed that every claw-free graph with $\delta \geq 4$ has a 2-factor. For the upper bound of the number of cycles in 2-factors, Broersma et al. [2] proved that a claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\max\left\{\frac{n-3}{\delta-1}, 1\right\}$ cycles. This upper bound is almost best possible. (See [12].) Faudree et al. [6] studied a pair of a maximum independent set and a 2-factor of a claw-free graph G which together dominate G and showed that if G is a claw-free graph with $\delta \geq \frac{2n}{\alpha} - 2$ and $n \geq \frac{3\alpha^3}{2}$, then for any maximum independent set S, G has a 2-factor with α cycles such that each cycle contains exactly one vertex in S. The following problems were posed in their article.

Conjecture A ([6]). *Let G be a claw-free graph.*

1. *If $\delta \geq \frac{n}{\alpha} \geq 5$, then there exist a maximum independent set S and a 2-factor with α cycles such that each cycle contains exactly one vertex of S.*
2. *If $\delta \geq \alpha + 1$, then for any maximum independent set S, there exists a 2-factor with α cycles such that each cycle contains exactly one vertex in S.*

In this paper, we study 2-factors that just divide a given maximum independent set S, i.e., we require that every cycle contains at least one vertex of S, and so the number of cycles in a 2-factor can be smaller than α. The original question was

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posed by Kaiser when the third author gave a lecture at the University of West Bohemia. However, in general still we need the condition $\delta \geq n/\alpha$ because for any positive integer δ with $\frac{n}{\alpha} - \frac{1}{2\delta} < \delta < \frac{n}{\alpha}$, there exists an infinite family of line graphs with minimum degree δ every 2-factor of which contains more than α cycles (see [6,12]). However 2-connectivity decreases the lower bound of minimum degrees. Our main result of this paper is the following.

Theorem 1. *If G is an l-connected claw-free graph with $\delta \geq 3$ and $l \in \{2, 3\}$, then for any maximum independent set S, G has a 2-factor such that each cycle contains at least $l - 1$ vertices in S.*

We will show this in Section 2. Since a 3-connected claw-free graph has a 2-factor in which each cycle contains at least two vertices in a given maximum independent set by Theorem 1, the number of the cycles in the 2-factor is at most $\frac{\alpha}{2}$. It is well known that the independence number of a claw-free graph is at most $\frac{2n}{\delta+2}$ (for instance, see [6]), and so we obtain the following.

Corollary 2. *A 3-connected claw-free graph has a 2-factor with at most $\frac{\alpha}{2} \leq \frac{n}{\delta+2}$ cycles.*

This improves the upper bound $\frac{n-3}{\delta-1}$ given by Broersma et al. in [2] if the connectivity is at least three and also the upper bound $\frac{2n}{15}$ given by Jackson and Yoshimoto in [9] if $\delta \geq 14$.

Finally we give some additional definitions and notation. A subgraph D is *dominating* a graph G if $G - V(D)$ is edgeless. The degree of a vertex u in G is denoted by $d_G(u)$ and we denote the set of all the vertices of degree at least k in G by $V_{\geq k}(G)$, and $V_k(G) = \{d_G(u) = k \mid u \in G\}$. The *edge-degree* of an edge xy is defined as $d_G(x) + d_G(y) - 2$. An edge subset E_0 is called *independent* if no pair of edges in E_0 are adjacent. We denote the subgraph induced by the vertex set of a subgraph B in G by $G[B]$. A graph G with at least $k + 1$ edges is *essentially* k-edge-connected if for any edge set E_0 of at most $k - 1$ edges, $G \setminus E_0$ contains at most one component with edges.

2. Proof of Theorem 1

Let G_0 be an l-connected claw-free graph with $\delta \geq 3$ and $l \in \{2, 3\}$ and S_0 be any maximum independent set of G_0. We look for a 2-factor in G_0 in which each cycle contains at least $l - 1$ vertices in S_0.

We use Ryjáček closure of a claw-free graph G which is defined as follows: for each vertex x of G, $N_G(x)$ induces a subgraph $G[N_G(x)]$ with at most two components; otherwise there is an induced claw. If $G[N_G(x)]$ has two components, both of them must be cliques. In the case that $G[N_G(x)]$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$. The *closure* $cl(G)$ of G is a graph obtained by recursively repeating this operation, as long as this is possible. Ryjáček [10] showed that the closure $cl(G)$ is uniquely determined and G is Hamiltonian if and only if $cl(G)$ is Hamiltonian.

Ryjáček et al. [11, Theorem 4] proved that for any mutually vertex-disjoint cycles $D_1, \ldots, D_p$ in $cl(G)$, a claw-free graph G has mutually vertex-disjoint cycles $C_1, \ldots, C_q$ with $p \geq q$ such that $\bigcup_{i=1}^{p} V(D_i) \subseteq \bigcup_{j=1}^{q} V(C_j)$. By modifying the proof we can easily improve this result as follows:

Lemma 3. *If G is a claw-free graph and $D_1, \ldots, D_p$ are mutually vertex-disjoint cycles in $cl(G)$, then G has mutually vertex-disjoint cycles $C_1, \ldots, C_q$ with $p \geq q$ such that for each D_i, there exists C_j such that $V(D_i) \subseteq V(C_j)$.*

If $cl(G_0)$ has a 2-factor $\bigcup_{i=1}^{p} D_i$ in which each cycle D_i contains at least $l - 1$ vertices in S_0, then by the above lemma, G_0 has vertex-disjoint cycles $C_1, \ldots, C_q$ such that for each D_i, there exists C_j such that $V(D_i) \subseteq V(C_j)$. Since

$$|C_j \cap S_0| \geq |D_i \cap S_0| \geq l - 1 \quad \text{and} \quad \bigcup_{i=1}^{p} V(D_i) = V(cl(G_0)) = V(G_0),$$

$\bigcup_{j=1}^{q} C_j$ is a 2-factor of G_0 with the required properties. We rephrase moreover the above statement using the following result.

Lemma B (*Ryjáček [10]*)**.** *For any claw-free graph G, there exists a triangle-free graph H such that $L(H) = cl(G)$.*

Let H_0 be a triangle-free graph such that $L(H_0) = cl(G_0)$. By the above facts, for Theorem 1, it is sufficient to show that:

$L(H_0)$ has a 2-factor in which each cycle contains at least $l - 1$ vertices in S_0. (1)

Let H be a graph and $\mathcal{D}$ a set of mutually edge-disjoint closed trails and stars in H. If every star has at least three edges and every edge in $E(G) \setminus \bigcup_{D \in \mathcal{D}} E(D)$ is incident to a closed trail in $\mathcal{D}$, then $\mathcal{D}$ is called a *system that dominates* H. Gould and Hynds [7] showed that the line graph $L(H)$ has a 2-factor with c cycles if and only if there exists a system that dominates H with c elements. Hence, we look for a system that dominates H_0 such that the corresponding 2-factor of $L(H_0)$ satisfies (1).

The set in H_0 corresponding to S_0 is an edge set. We denote the edge set also by S_0. Notice that S_0 is independent in G_0, but S_0 is not always independent in $L(H_0) = cl(G_0)$, i.e.,

S_0 is possibly not independent in H_0.

In either case, the following claim implies (1) immediately.