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The maximum radius of graphs with given order and minimum degree

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ABSTRACT

Let G be a graph with order n and minimum degree $\delta(\geq 2)$. Erdős et al. found an upper bound of the radius r of G, which is $\frac{3}{2}\frac{n-3}{\delta+1}+5$. They noted that this bound is tight apart from the exact value of the additive constant. In this paper, when $r \geq 3$, we decrease this bound to $\lfloor \frac{3}{2}\frac{n}{\delta+1} \rfloor$, the extremal value.

1. Introduction

Klee and Quaife [4] found the minimum order of graphs with given diameter, connectivity and minimum degree. Especially for regular graphs, there are further results [1,2]. The extremal 3-regular graphs are classified except when they have connectivity 3 and even diameter [3,5,6]. Erdős et al. [7] gave an upper bound for the maximum diameter of graphs with given order n and minimum degree δ. They also proved that $\frac{3}{2}\frac{n-3}{\delta+1}+5$ is an upper bound of the radius r of the graph. They pointed out that this bound is sharp up to constant. In this paper, when $r \geq 3$, we decrease this upper bound to $\left\lfloor \frac{3}{2}\frac{n}{\delta+1} \right\rfloor$, the extremal value.

Let $G = (V, E)$ be a graph with radius $\mathrm{rad}(G) = r$. Assume that the minimum degree of G is δ. Let v_0 be the center of G. Then there is $v_0^* \in V$ such that $\mathrm{dist}(v_0, v_0^*) = r$. Define $S_i = \{y \in V | \mathrm{dist}(v_0, y) = i\}$ and let $a_i = |S_i|$ for $0 \leq i \leq r$. Further we define $S_{\leq j} = \cup_{0 \leq i \leq j} S_i$ and $S_{\geq j} = \cup_{j \leq i \leq r} S_i$. We also write the neighborhood of a vertex $v \in V$ by $N[v]$. Note that $|N[v]| \geq \delta + 1$ and $|V| \geq d(\delta + 1)$ if some elements $x_1, \ldots, x_d \in V$ have mutually disjoint neighborhoods.

2. Some lemmas

Lemma 1. *In S_i, there is a pair of vertices at distance at least 3 when $3 \leq i \leq r - 3$.*

Proof. Suppose there is $i (3 \leq i \leq r-3)$ such that $\mathrm{dist}(u, v) \leq 2$ for any $u, v \in S_i$. Choose $u_3 \in S_3$ such that there is $u \in S_i$ with $\mathrm{dist}(u_3, u) = i - 3$. For all $y \in S_{\geq i}$, $y \in S_{i+h}$ for some $0 \leq h \leq r - i$. Choose $v_y \in S_i$ such that $\mathrm{dist}(v_y, y) = h$. We have

$$\mathrm{dist}(u_3, y) \leq \mathrm{dist}(u_3, u) + \mathrm{dist}(u, v_y) + \mathrm{dist}(v_y, y) \leq i - 3 + 2 + h \leq r - 1.$$

For all $y \in S_{\leq i-1}$, we have

$$\mathrm{dist}(u_3, y) \leq \mathrm{dist}(u_3, v_0) + \mathrm{dist}(v_0, y) = 3 + i - 1 \leq r - 1.$$

This is a contradiction. □

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Lemma 2. *For any $3 \le i \le r-3$, we have $a_{i-1}+a_i+a_{i+1} \ge 2(\delta+1)$.*

Proof. By Lemma 1, there is a pair $u_1, u_2 \in S_i$ such that $\mathrm{dist}(u_1, u_2) \ge 3$. Since $N[u_1], N[u_2] \subset S_{i-1} \cup S_i \cup S_{i+1}$ and $N[u_1] \cap N[u_2] = \emptyset$, $a_{i-1}+a_i+a_{i+1} \ge |N[u_1] \cup N[u_2]| \ge 2(\delta+1)$. □

Lemma 3. 1. *If there are $x, y \in S_2$ with $\mathrm{dist}(x, y) = 4$, then $2a_0+2a_1+2a_2+a_3 \ge 4(\delta+1)$.*
2. *If there are $x, y \in S_{r-2}$ with $\mathrm{dist}(x, y) \ge 4$ and there are $x', y' \in S_{r-1}$ such that $\{x, x'\}, \{y, y'\} \in E$, then $2a_r+2a_{r-1}+2a_{r-2}+a_{r-3} \ge 4(\delta+1)$.*

Proof. 1. Since $x, y \in S_2$, there are $v_1, w_1 \in S_1$ such that $\{v_1, x\}, \{w_1, y\} \in E$. Since $\mathrm{dist}(x, y) = 4$, we have $\mathrm{dist}(x, w_1) \ge 3$ and $\mathrm{dist}(y, v_1) \ge 3$. So for any $z \in S_0 \cup S_1 \cup S_2$, z is contained in at most two of the four sets $N[v_1], N[w_1], N[x]$ and $N[y]$. And for any $z' \in S_3$, z' is contained in at most one of the four sets $N[v_1], N[w_1], N[x]$ and $N[y]$. We have $2(a_0+a_1+a_2)+a_3 \ge 4(\delta+1)$.
2. The proof of (2) is similar to that of (1). □

Since $\mathrm{dist}(v_0, v_0^*) = r$, there is a path $v_0 \to v_1 \to \cdots \to v_r = v_0^*$ with $v_i \in S_i (0 \le i \le r)$. Since $\mathrm{rad}(G) = r$, there is $v_2^* \in V$ such that $\mathrm{dist}(v_2, v_2^*) = r$. Let $\mathrm{dist}(v_0, v_2^*) = s$. Then there is a path $v_0 = w_0 \to w_1 \to \cdots \to w_s = v_2^*$ with $w_i \in S_i (0 \le i \le s)$. Clearly, $w_s \in S_{\ge r-2}$. If $w_s \in S_{r-2}$, then $\mathrm{dist}(v_2, w_2) = 4$ and $\mathrm{dist}(v_{r-2}, w_s) \ge 4$.

Lemma 4. *Assume $w_s \in S_{r-2}$ $(r \ge 4)$. If $\mathrm{dist}(v_i, w_j) \le 2$ for some $2 \le i \le r$ and $0 \le j \le r-2$, then $j = i-2$.*

Proof. If $j < i-2$, then

$$r = \mathrm{dist}(v_0, v_r) \le \mathrm{dist}(v_0, w_j) + \mathrm{dist}(w_j, v_i) + \mathrm{dist}(v_i, v_r) \le j+2+r-i < r,$$

a contradiction. If $j > i-2$, then

$$r = \mathrm{dist}(v_2, w_s) \le \mathrm{dist}(v_2, v_i) + \mathrm{dist}(v_i, w_j) + \mathrm{dist}(w_j, w_s) \le i-2+2+s-j < r,$$

a contradiction. So $j = i-2$. □

3. Main theorems

Theorem 1. *Let G be a graph with radius $r(\ge 3)$ and minimum degree $\delta(\ge 2)$. Then the order n of G satisfies*

$$n \ge \frac{2}{3} r(\delta+1).$$

Proof. If $r = 3k$, then since $a_0+a_1 \ge \delta+1$, $a_{r-1}+a_r \ge \delta+1$ and by Lemma 2,

$$|V| = \sum_{i=0}^{r} a_i = a_0 + a_1 + \sum_{i=1}^{k-1} (a_{3i-1}+a_{3i}+a_{3i+1}) + a_{r-1} + a_r \ge 2k(\delta+1) = \frac{2}{3} r(\delta+1).$$

Consider the case where $r = 3k+1$. Suppose that $w_s \in S_{r-1} \cup S_r$. Then $\mathrm{dist}(v_{r-1}, w_s) \ge \mathrm{dist}(w_s, v_2) - \mathrm{dist}(v_{r-1}, v_2) = r-(r-3) = 3$. Since $N[v_{r-1}] \cap N[w_s] = \emptyset$, $a_r + a_{r-1} + a_{r-2} \ge 2(\delta+1)$. We have

$$\begin{aligned} |V| &= \sum_{i=0}^{r} a_i = a_0 + a_1 + \sum_{i=1}^{k-1} (a_{3i-1}+a_{3i}+a_{3i+1}) + a_{r-2} + a_{r-1} + a_r \\ &\ge (2k+1)(\delta+1) > \left(2k+\frac{2}{3}\right)(\delta+1) = \frac{2}{3} r(\delta+1). \end{aligned}$$

Assume that $w_s \in S_{r-2}$. Then $\mathrm{dist}(v_2, w_2) = 4$ and $\mathrm{dist}(v_{r-2}, w_s) \ge 4$. If $\mathrm{dist}(v_r, w_s) = 2$, then there is a vertex $x \in S_{r-1}$ such that $\{w_s, x\} \in E$. So by Lemma 3,

$$2(a_0+a_1+a_2)+a_3 \ge 4(\delta+1)$$

and

$$2(a_r+a_{r-1}+a_{r-2})+a_{r-3} \ge 4(\delta+1).$$

By Lemma 2,

$$\sum_{i=1}^{k-1} (a_{3i-1}+a_{3i}+a_{3i+1}) \ge 2(k-1)(\delta+1),$$

$$\sum_{i=1}^{k-2} (a_{3i+1}+a_{3i+2}+a_{3i+3}) \ge 2(k-2)(\delta+1)$$