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Face vectors of subdivided simplicial complexes

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ABSTRACT

Brenti and Welker have shown that, for any $(d-1)$-dimensional simplicial complex X, the f-vectors of successive barycentric subdivisions of X have roots which converge to fixed values depending only on the dimension of X. We improve and generalize this result here. We begin with an alternative proof based on geometric intuition. We then prove an interesting symmetry of these roots about the real number -2. This symmetry can be seen via a nice algebraic realization of barycentric subdivision as a simple map on formal power series in two variables. Finally, we use this algebraic machinery with some geometric motivation to generalize the combinatorial statements to arbitrary subdivision methods: any subdivision method will exhibit similar limit behavior and symmetry. Our techniques allow us to compute explicit formulas for the values of the limit roots in the case of barycentric subdivision.

1. Introduction

Throughout this paper, we let X be an arbitrary finite simplicial complex of dimension $d-1$, and we assume that all vectors and matrices will be indexed by rows and columns starting at 0. We are interested in roots of the *f-polynomial* of X, defined as follows. Let f_i^X denote the number of i-dimensional faces of X. We declare that $f_{-1}^X = 1$, where the (-1)-dimensional face is the empty face, $\emptyset$. The *face vector*, or *f-vector*, of X is the vector

$$f^X := \left(f_{-1}^X, f_0^X, \ldots, f_{d-1}^X\right).$$

Let $\mathbf{t}$ denote the column vector of powers of t, $(t^d, t^{d-1}, \ldots, t^0)^T$. The *$f$-polynomial* $f^X(t)$ encodes the f-vector as a polynomial:

$$f^X(t) := \sum_{j=0}^{d} f_{j-1}^X t^{d-j} = f^X \mathbf{t}.$$

Much work has been devoted to the study of f-vectors of simplicial complexes, their close relatives, the g- and h-vectors, and the associated polynomials. As it turns out, the entries of these objects encode many combinatorial and algebraic aspects of the complex to which they are associated (see [1,3,9] for background and further references).

We focus on a recent result of Brenti and Welker which may initially appear surprising. Let X' denote the barycentric subdivision of X, and more generally let $X^{(n)}$ denote the nth barycentric subdivision of X.

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Theorem 1.1 ([2]). *Let X be a $(d-1)$-dimensional simplicial complex. As n grows, the $d-1$ largest roots of $f^{X^{(n)}}(t)$ converge to $d-1$ negative real numbers which depend only on d, not on X.*

We provide some geometric intuition and motivation for why this result holds. We offer an alternate proof of this theorem based on these geometric observations. In the process, we show how to compute the $d-1$ real values for each d. Our first main theorem is the following.

Theorem A. *Let X be a $(d-1)$-dimensional simplicial complex. Then the $d-1$ largest roots of $f^{X^{(n)}}(t)$ converge to $d-1$ values which are the roots of a polynomial $p_d(t)$, depending only on d, whose coefficients are listed in the last row of the inverse of a particular matrix, P_d.*

The entries of the matrix P_d, and of its inverse P_d^{-1}, are computed in Section 6. Our calculations allow us to compute the 'limit roots' thus obtained. In the examples, we observed that these 'limit roots' are symmetrically distributed about the point -2, with respect to the Möbius transformation $x \mapsto \frac{-x}{x+1}$. Our second main theorem proves this symmetry.

Theorem B. *For any dimension $(d-1)$, the $d-1$ 'limit roots' are invariant under the map $x \mapsto \frac{-x}{x+1}$.*

In fact, more can be said. The existence of a 'limit polynomial' and the symmetry result hold for an arbitrary subdivision method, as we show in Theorem 5.5.

For barycentric subdivision, this symmetry can be seen through a beautiful algebraic theorem. Barycentric subdivision, considered as a map on f-polynomials, induces a function $b : \mathbb{Z}[t] \to \mathbb{Z}[t]$, as in Section 4. We list the values of b on monomials as coefficients in the formal power series in the variable x over $\mathbb{Z}[t]$, by defining $B : \mathbb{Z}[t][[x]] \to \mathbb{Z}[t][[x]]$ by $B\left(\sum_{k\geq 0} g_k(t)x^k\right) := \sum_{k\geq 0} b\big(g_k(t)\big)x^k$.

Theorem C. *In $\mathbb{Z}[t][[x]]$, barycentric subdivision satisfies the identity*

$$B\big(e^{tx}\big) = \frac{1}{1-\big(e^x-1\big)t}.$$

This paper is organized as follows. In Section 2, we discuss the geometric intuition and motivation behind Theorem 1.1. In Section 3, we prove Theorem A. In Section 4, we prove the symmetry stated in Theorem B, and prove Theorem C. In Section 5, we extend the symmetry to arbitrary subdivision methods. We end with Section 6, where we compute the entries in the inverse matrix P_d^{-1} found in Theorem A as well as all limit polynomials and roots up to the value $d = 10$.

2. Geometric motivation

Brenti and Welker's theorem may be surprising at first: there is no dependence on the initial complex X, only on the dimension $d-1$. However, geometrically this makes perfect sense. Barycentrically subdividing a simplicial complex X over and over again causes the resulting complex $X^{(n)}$ to have far more cells than the original X. Because higher-dimensional cells contribute more new cells (in every dimension) upon subdividing than lower-dimensional ones, the top-dimensional cells begin to dominate in their number of contributions to subdivisions. For example, think of geometric realizations so that $X^{(n)}$ is a subset of X. Then a randomly chosen cell of $X^{(n)}$ should, with higher and higher probability as n increases, be contained in the interior of a top-dimensional cell of X, as top-dimensional cells contribute far more cells to $X^{(n)}$ than other cells.

Each of the f_{d-1}^X top-dimensional cells of X contributes the same number of cells to $X^{(n)}$. Since these cells eventually dominate contributions from smaller-dimensional cells, the f-polynomial for $X^{(n)}$ can be approximated by f_{d-1}^X times the f-polynomial for the nth barycentric subdivision $\sigma_d^{(n)}$ of a single top-dimensional cell σ_d. Since the roots of a polynomial are unaffected by multiplication by constants, the roots of $f^{X^{(n)}}(t)$ converge to the roots of $f^{\sigma_d^{(n)}}(t)$ as n increases.

By definition, the coefficients of $f^{\sigma_d^{(n)}}(t)$ record the number of cells of each dimension occurring in $\sigma_d^{(n)}$. The number of cells in each dimension is bounded by a constant times the number of top-dimensional cells. Thus, if we normalize $f^{\sigma_d^{(n)}}(t)$ by dividing by the number of top-dimensional cells, we have coefficients which, for each k, record the density of k-cells relative to the number of top-dimensional cells. As this density is positive but strictly decreases upon subdividing, there is a limiting value for the coefficient. Thus, there is a limiting polynomial, with well-defined roots.

We now formalize this intuition.

3. *f*-polynomials of barycentric subdivisions

3.1. Barycentric subdivision and the matrix Λ_d

To prove Theorem A, we begin by observing the effect of barycentric subdivision on f-vectors. One key observation is that barycentric subdivision multiplies f-vectors by a fixed matrix, Λ_d, defined as follows.