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Further results on optimal $(v, 4, 2, 1)$-OOCs

Xiaomiao Wang, Yanxun Chang *

Institute of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

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ABSTRACT

Let $\Phi(v, k, \lambda_a, \lambda_c)$ denote the maximum possible size among all $(v, k, \lambda_a, \lambda_c)$-OOCs. A $(v, k, \lambda_a, \lambda_c)$-OOC is said to be *optimal* if its size is equal to $\Phi(v, k, \lambda_a, \lambda_c)$. In this paper, the constructions and the sizes of optimal $(v, 4, 2, 1)$-OOCs are investigated. An upper bound for $\Phi(v, 4, 2, 1)$ is improved. The exact value of $\Phi(v, 4, 2, 1)$ with $v \le 201$ is given with the aid of computer search. An optimal $(24hv, 4, 2, 1)$-OOC with $h \in \{1, 2\}$ and $v = p_1p_2 \cdots p_r$, where prime $p_i \equiv 1 \pmod 6$ is constructed recursively. The existence of g-regular $(gp, 4, 2, 1)$-OOCs for $g = 3, 6, 9, 16$, and p a prime satisfying a suitable congruence is established by direct constructions. Furthermore, the sizes of several new infinite classes of optimal $(v, 4, 2, 1)$-OOCs are obtained. In particular, $\Phi(v, 4, 2, 1) = U(v)$ for positive integer $v \equiv 80, 400 \pmod{480}$.

1. Introduction

A $(v, k, \lambda_a, \lambda_c)$ *optical orthogonal code* (or $(v, k, \lambda_a, \lambda_c)$-OOC) can be viewed as a collection $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$ of k-subsets (codeword-sets) of Z_v, such that any two distinct translates of a codeword-set share at most λ_a elements, while any two translates of two distinct codeword-sets share at most λ_c elements:

(1) (*the auto-correlation property*) $|C_i \cap (C_i + t)| \le \lambda_a$ for $1 \le i \le s$ and $1 \le t \le v - 1$;

(2) (*the cross-correlation property*) $|C_i \cap (C_j + t)| \le \lambda_c$ for $1 \le i \ne j \le s$ and $0 \le t \le v - 1$.

The number of codeword-sets is called the *size* of $\mathcal{C}$. Let $\Phi(v, k, \lambda_a, \lambda_c)$ denote the maximum possible size among all $(v, k, \lambda_a, \lambda_c)$-OOCs. A $(v, k, \lambda_a, \lambda_c)$-OOC is said to be *optimal* if its size is equal to $\Phi(v, k, \lambda_a, \lambda_c)$. When $\lambda_a = \lambda_c = \lambda$, the notations of (v, k, λ)-OOC and $\Phi(v, k, \lambda)$ are employed. From the definition of optimal OOC, it is straightforward that an optimal OOC exists for all parameter values: it is simply an OOC of the largest possible size. For given parameters v, k, λ_a and λ_c, the constructions and determining the sizes of optimal $(v, k, \lambda_a, \lambda_c)$-OOCs are apparently difficult tasks. It is well known that the size of a (v, k, λ)-OOC cannot exceed the Johnson bound (see [26]), that is,

$$\Phi(v, k, \lambda) \le J(v, k, \lambda)$$

where $J(v, k, \lambda) = \lfloor \frac{1}{k} \lfloor \frac{v-1}{k-1} \lfloor \cdots \lfloor \frac{v-\lambda}{k-\lambda} \rfloor \cdots \rfloor \rfloor \rfloor$.

Optical orthogonal codes have many important applications because of their good correlation properties. This was first motivated by an application in a fiber-optic code-division multiple-access (CDMA) channel. Recent work has been done on using OOCs for multimedia transmission in fiber-optic local-area networks (LANs) and in multi-rate fiber-optic CDMA systems. The reader may refer to [23,24,28,33] for details.

* Corresponding author.
E-mail address: yxchang@bjtu.edu.cn (Y. Chang).

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For a long time, research on OOCs has mainly concentrated on the case $\lambda_a = \lambda_c = \lambda$ in the literature. Constructions of optimal $(v, k, 1)$-OOCs has been investigated in [8,23]. It is often closely related to *difference families* [1,7,13,20,21,35] and to *relative difference families* [9,10,15–19,27,31] in view of their importance in design theory [2,6]. Recently, several results have also been done on optimal (v, k, λ)-OOCs with $\lambda = 2$ [3,22,25] and even with $\lambda > 2$ [4].

There is little in the literature regarding optimal $(v, k, \lambda_a, \lambda_c)$-OOCs with $\lambda_a \neq \lambda_c$. An earlier investigation about them was given in [34], where it is shown that the size of a $(v, k, \lambda_a, \lambda_c)$-OOC cannot exceed $\frac{\lambda_a(v-1)(v-2)\cdots(v-\lambda_c)}{k(k-1)(k-2)\cdots(k-\lambda_c)}$ with $\lambda_a \geq \lambda_c$. Known results on optimal $(v, k, k-1, 1)$-OOCs (which are also called *conflict-avoiding codes*) have been treated in [29,30]. The research on optimal $(v, 4, 2, 1)$-OOCs was first dealt with in [32], where several direct and recursive constructions for optimal $(v, 4, 2, 1)$-OOCs were presented. Later in [14], the constructions of perfect $(v, 4, 2, 1)$-OOCs were given for several infinite classes. In [5], all optimal $(v, 4, 2, 1)$-OOCs with $v < 71$ are determined up to equivalence.

In this paper, we shall investigate the sizes and constructions of optimal $(v, 4, 2, 1)$-OOCs. In the next section, we introduce the concept of good difference matrix and state recursive constructions for $(v, 4, 2, 1)$-OOCs. In Section 3, we improve the upper bound of the sizes of optimal $(v, 4, 2, 1)$-OOCs, and we also give direct constructions of optimal $(v, 4, 2, 1)$-OOCs for $v \leq 201$ with the aid of computer. In Section 4, we construct optimal $(24hv, 4, 2, 1)$-OOCs with $h \in \{1, 2\}$ and $v = p_1p_2\cdots p_r$, where prime $p_i \equiv 1 \pmod 6$. Finally, in Section 5, we establish the existence of a g-regular $(gp, 4, 2, 1)$-OOC with $g = 3, 6, 9, 16$, and obtain three new infinite classes of optimal $(v, 4, 2, 1)$-OOCs.

2. Recursive constructions

Let $K = (v_1, v_2, v_3)$-v_4 be a kite, that is, the graph consisting of a triangle $\{v_1, v_2, v_3\}$ with an attached edge $\{v_3, v_4\}$.

An $(n, K, 1)$ *difference matrix*, briefly DM$(n, K, 1)$, is a $4 \times n$ matrix with entries in Z_n, such that the differences between its i-th row and its j-th row is a permutation of Z_n whenever v_i and v_j are adjacent in K. (For the definition of an (n, Γ, λ) difference matrix where Γ is any graph and λ is any positive integer, the reader is referred to [11]). A DM$(n, K, 1)$ is called *good* if it is of the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1+b_1 & a_2+b_2 & \dots & a_n+b_n \\ 0 & 0 & \dots & 0 \\ 2a_1+b_1 & 2a_2+b_2 & \dots & 2a_n+b_n \end{pmatrix}.$$

The following result can be found in [32].

Lemma 2.1 *([32]). If gcd $(n, 6) = 1$, then*

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 2\cdot 1 & 2\cdot 2 & \dots & 2\cdot n \\ 0 & 0 & \dots & 0 \\ 3\cdot 1 & 3\cdot 2 & \dots & 3\cdot n \end{pmatrix}$$

is a good $(n, K, 1)$ difference matrix.

In order to state Lemmas 2.2 and 2.3, we recall some basic terminologies defined in [14].

For a given subset C of Z_v, its *list of differences* is denoted by ΔC, that is the multiset of all differences $x - y$ with (x, y) an ordered pair of distinct elements of C. The underlying set of ΔC is called the *support* of ΔC and it is denoted by supp(ΔC). More generally, the list of differences of a set $\mathcal{C}$ of subsets of Z_v is the multiset $\Delta\mathcal{C} = \bigcup_{C\in\mathcal{C}} \Delta C$. Also, the support of $\Delta\mathcal{C}$ is the underlying set of $\bigcup_{C\in\mathcal{C}} \Delta C$.

If δ is the size of supp(ΔC), namely the number of distinct differences appearing in ΔC, we will say that C is of *type* $t(C) = \delta$. It is obvious that $t(C)$ cannot exceed the size of ΔC, that is $t(C) \leq k(k-1)$ where k is the size of C. Saying that $\mathcal{C}$ is of type $[\delta_1^{x_1}, \delta_2^{x_2}, \dots, \delta_t^{x_t}]$, we mean that $\mathcal{C}$ has exactly x_i codeword-sets of type δ_i for $i = 1, 2, \dots, t$.

The *missing differences* of a $(v, k, \lambda_a, 1)$-OOC $\mathcal{C}$ is the set M of Z_v that are not covered by the list of differences of $\mathcal{C}$. If the OOC is of type $[\delta_1^{x_1}, \delta_2^{x_2}, \dots, \delta_t^{x_t}]$, it is clear that $m = |M| = v - (\delta_1x_1 + \delta_2x_2 + \cdots + \delta_tx_t)$. If M is a subgroup of Z_v, then the OOC is called *m-regular*.

The following constructions come from Theorem 5.3 of [32], here we rewrite it with two lemmas using the concept of good difference matrix.

Lemma 2.2. *Suppose that there exist both a g-regular $(gm, 4, 2, 1)$-OOC $\mathcal{F}$ such that $|\text{supp}(\Delta X)| = 8$ for every $X \in \mathcal{F}$, and a good* DM$(n, K, 1)$. *Then there exists a gn-regular $(gmn, 4, 2, 1)$-OOC $\mathcal{F}'$ satisfying $|\text{supp}(\Delta Y)| = 8$ for $Y \in \mathcal{F}'$.*

Lemma 2.3. *If there are a gm-regular $(gmn, 4, 2, 1)$-OOC $\mathcal{F}$ such that $|\text{supp}(\Delta X)| = 8$ for every $X \in \mathcal{F}$, and a $(gm, 4, 2, 1)$-OOC $\mathcal{F}'$ with size s, then there exists a $(gmn, 4, 2, 1)$-OOC with size $gm(n-1)/8 + s$ that is g-regular if $\mathcal{F}'$ is such. Moreover, every codeword-set Y of the resultant OOC satisfies $|\text{supp}(\Delta Y)| = 8$ if $\mathcal{F}'$ has the same property.*