Discrete Mathematics 312 (2012) 341–350

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The existence spectrum for $(3, \lambda)$-GDDs of type $g^t u^1$ ☆

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ARTICLE INFO

Article history:
Received 1 January 2011
Received in revised form 29 August 2011
Accepted 20 September 2011
Available online 26 October 2011

Keywords:
Group divisible design
Wilson's fundamental construction
Cyclic partial triple system

ABSTRACT

The necessary and sufficient conditions for the existence of a $(3, 1)$-GDD of type $g^t u^1$ have been established by Colbourn et al. In this paper, the existence of a $(3, \lambda)$-GDD of type $g^t u^1$ for any $\lambda \geq 2$ is investigated. Finally, its existence spectrum is completely determined.

1. Introduction

Let X be a finite set of v elements, and let K be a set of positive integers and λ a positive integer. A *group divisible design* of index λ and order v, denoted (K, λ)-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of X into subsets (called *groups*); (2) $\mathcal{B}$ is a collection of subsets of X (called *blocks*), each of cardinality from K, such that a group and a block contain at most one common element; (3) every pair of elements from distinct groups occurs in exactly λ blocks. If $\mathcal{G}$ contains t_i groups of size g_i for $1 \leq i \leq s$, then we call $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ the *group type* (or *type*) of the GDD. The notation (k, λ)-GDD is usually employed if $K = \{k\}$. A $(k, 1)$-GDD of type t^k is called a *transversal design*, and is denoted by TD(k, t). A $(3, \lambda)$-GDD of type 1^v is commonly called a *triple system* of index λ and order v, or TS(v, λ). A TS$(v, 1)$ is called a *Steiner triple system* of order v, and is denoted by STS(v).

Let α be a positive integer. An α-*parallel class* of a set X is a collection of subsets of X containing every element of X exactly α times. A (K, λ)-GDD is α-*resolvable* if its blocks can be partitioned into α-parallel classes. A 1-parallel class is called a *parallel class* and a 1-resolvable GDD is called *resolvable*. It is well known that a resolvable TD$(3, t)$, or RTD$(3, t)$, exists if and only if $t \neq 2, 6$. A resolvable STS(v) is called a *Kirkman triple system*, and is denoted by KTS(v). A KTS(v) exists if and only if $v \equiv 3 \pmod 6$ (see [7]).

In this paper, we focus on the existence of a $(3, \lambda)$-GDD of type $g^t u^1$. The main result is stated as follows.

Theorem 1.1 (*Main Theorem*). *Let λ be a positive integer, and let g, t, and u be nonnegative integers. There exists a $(3, \lambda)$-GDD of type $g^t u^1$ if and only if all of the following conditions are satisfied:*

(1) *if $g > 0$, then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*
(2) *$u \leq g(t-1)$ or $gt = 0$;*
(3) *$\lambda g(t-1) + \lambda u \equiv 0 \pmod 2$ or $gt = 0$;*
(4) *$\lambda gt \equiv 0 \pmod 2$ or $u = 0$;*
(5) *$\lambda g^2 t(t-1)/2 + \lambda gtu \equiv 0 \pmod 3$.*

☆ This work was supported by National Natural Science Foundation of China under grant 61071221 and the Fundamental Research Funds for the Central Universities.
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0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.09.024

We first prove the necessity of the Main Theorem. If $t = 2$, we fix an element x in the group of size g. The number of pairs containing x is $\lambda(g+u)$. On the other hand, there are λg blocks containing x, and hence the number of pairs containing x is $2\lambda g$. It follows that $\lambda(g+u) = 2\lambda g$, which amounts to $u = g$. Thus (1) holds. Fix an element x in the group of size g. There are in total λu blocks which contain both x and an element in the group of size u; the other λu elements in these blocks are obviously in the group of size g. Note that there are in total $\lambda g(t-1)$ pairs containing both x and an element in the group of size g; the inequality $u \leq g(t-1)$ in (2) then follows if $gt \neq 0$. For any fixed element x in a group of size g, the number of blocks containing x is $(\lambda g(t-1) + \lambda u)/2$, which should be an integer. Thus $\lambda g(t-1) + \lambda u \equiv 0 \pmod 2$ if $gt \neq 0$, i.e. condition (3) follows. Similarly, by considering the number of blocks containing an element in the group with size u, the congruence equality (4) follows. Since the total number of blocks is $(\lambda g^2 t(t-1)/2 + \lambda gtu)/3$, condition (5) follows.

We call a quadruple (g, t, u, λ) *admissible* provided that the five conditions in the Main Theorem all hold. In the remainder of this paper, we mainly establish the sufficiency of the Main Theorem. The sufficiency when $\lambda = 1$ has been proved by Colbourn et al. in [4]. The necessary conditions for any index λ and $u = 0$ or $u = g$ have also been shown to be sufficient; see for instance [12]. The case of $g = 1$ has also been handled; it is implied by the existence of incomplete triple systems. We sum up these known results in a theorem.

Theorem 1.2 ([3,4,12])**.** *The Main Theorem holds for any admissible (g, t, u, λ), where (1) $\lambda = 1$, or (2) $\lambda \geq 1$ and $u = 0$, or (3) $\lambda \geq 1$ and $g = 1$.*

By Theorem 1.2, we can always assume that g and u are all positive, $g \neq u$, and $t \geq 3$ when considering the existence of a $(3, \lambda)$-GDD of type $g^t u^1$. Section 2 displays several recursive and direct constructions for $(3, \lambda)$-GDDs. Section 3 deals with the existence of a $(3, 2)$-GDD of type $g^t u^1$, where $g \equiv 0 \pmod 3$. In Section 4, further constructions utilizing cyclic partial triple systems are presented to solve the case when u is relatively large. Finally, in Sections 5 and 6, we solve the cases $\lambda = 2, 3, 6$ and determine the existence spectrum for $(3, \lambda)$-GDDs of type $g^t u^1$.

2. Recursive and direct constructions

First, we have several classical recursive constructions such as Wilson's fundamental construction and filling-in-group constructions.

Lemma 2.1 ([11] Wilson's Fundamental Construction, WFC)**.** *Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a (K, λ)-GDD (called the master GDD), and let $w : X \to Z^+ \cup \{0\}$ be a weight function. For every block $B \in \mathcal{B}$, suppose that there is a $(3, \mu)$-GDD (called the ingredient GDD) of type $\{w(x) : x \in B\}$. Then there exists a $(3, \lambda\mu)$-GDD of type $\left\{\sum_{x\in G} w(x) : G \in \mathcal{G}\right\}$.*

Lemma 2.2 (Filling Construction I)**.** *Let $g \equiv 0 \pmod s$. Suppose that there exist both a (K, λ)-GDD of type $g^t u^1$ and a (K, λ)-GDD of type $(g/s)^s w^1$. Then there exists a (K, λ)-GDD of type $(g/s)^{st}(w+u)^1$.*

Proof. Place copies of the second GDD on the groups of the first, together with w additional points forming a group in each copy. □

Corollary 2.3. *Let $\lambda = 2, 3$, and let $t \geq 4$ be even. If there exists a $(3, \lambda)$-GDD of type $(2g)^{t/2}u^1$, then so does a $(3, \lambda)$-GDD of type $g^t(g+u)^1$.*

Proof. For $\lambda = 2, 3$, a $(3, \lambda)$-GDD of type g^3 exists by Theorem 1.2. Since a $(3, \lambda)$-GDD of type $(2g)^{t/2}u^1$ exists, the conclusion then follows by Filling Construction I. □

Lemma 2.4 (Filling Construction II)**.** *Let $u = sg + x$. Suppose that there exist both a (K, λ)-GDD of type $g^t u^1$ and a (K, λ)-GDD of type $g^s x^1$. Then there exists a (K, λ)-GDD of type $g^{t+s}x^1$.*

Proof. Place a copy of the second GDD on the group of size u of the first GDD to produce the desired design. □

Let X be a set of gt points and K a set of positive integers. A *modified group divisible design* of index λ, or (K, λ)-MGDD, is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of X into t g-subsets (called *groups*), and $\mathcal{H}$ is a partition of X into g t-subsets (called *holes*); (2) $|G \cap H| = 1$ for each $G \in \mathcal{G}$ and each $H \in \mathcal{H}$; (3) $\mathcal{B}$ is a collection of subsets of X (called *blocks*), each of cardinality from K, such that a block contains no more than one point of any group and any hole, and every pair of points from distinct groups and distinct holes occurs in exactly λ blocks. A $(\{3\}, \lambda)$-MGDD with t groups and g holes is usually denoted by $(3, \lambda)$-MGDD(g, t).

Lemma 2.5 ([1])**.** *The necessary and sufficient conditions for the existence of a $(3, \lambda)$-MGDD(g, t) are $g, t \geq 3$, $\lambda(g-1)(t-1) \equiv 0 \pmod 2$, and $\lambda gt(g-1)(t-1) \equiv 0 \pmod 6$.*

Lemma 2.6. *Let $(X, \mathcal{G}, \mathcal{B})$ be a $(3, \lambda)$-GDD. Suppose that there exists a $(3, \lambda)$-GDD of type $|G|^t u^1$ for each $G \in \mathcal{G}$, where $t \geq 3$. Then there exists a $(3, \lambda)$-GDD of type $|X|^t u^1$.*

Proof. We form the required GDD with group set $\{X \times \{i\} : i \in I_t\} \cup \{U\}$, where $|U| = u$. For each block $B \in \mathcal{B}$, place on $B \times I_t$ a $(3, 1)$-MGDD$(t, 3)$ (whose existence is assured by Lemma 2.5) with group set $\mathcal{G}_B = \{\{x\} \times I_t, x \in B\}$ and hole set $\{\{x, y, z\} \times \{i\} : i \in I_t\}$. Next, for each $G \in \mathcal{G}_B$, place on $(G \times I_t) \cup U$ a $(3, \lambda)$-GDD of type $|G|^t u^1$ with groups $G \times \{i\}$, $i \in I_t$, and U. The resultant is a $(3, \lambda)$-GDD of type $|X|^t u^1$. □