Discrete Mathematics 311 (2011) 577–581

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A generalization of plexes of Latin squares

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ARTICLE INFO

Article history:
Received 4 August 2010
Received in revised form 2 December 2010
Accepted 5 January 2011
Available online 4 February 2011

Keywords:
Latin square
Transversal
Duplex
k-plex

ABSTRACT

A k-plex of a Latin square is a collection of cells representing each row, column, and symbol precisely k times. The classic case of $k = 1$ is more commonly known as a *transversal*. We introduce the concept of a k-weight, an integral weight function on the cells of a Latin square whose row, column, and symbol sums are all k. We then show that several non-existence results about k-plexes can been seen as more general facts about k-weights and that the weight analogues of several well-known existence conjectures for plexes actually hold for k-weights.

1. Introduction and background

We call an integral weight function on the cells of a Latin square a *k-weight* if the sum over each row, column, and symbol is k. The primary purpose of this paper is to show that several important non-existence results and existence conjectures concerning k-plexes hold in the much weaker setting of k-weights. Our hope, therefore, is that k-weights may prove to be a useful generalization for better understanding k-plexes.

In Section 2 we establish a simple lemma that is employed in several of our arguments. In Section 3 we generalize a non-existence result of Wanless for odd-plexes to show that certain Latin squares have no odd-weights. We also give a construction to show that analogues of conjectures of Ryser and Rodney about k-plexes hold for k-weights. In Section 4 we generalize recent results of Stein and Szabó concerning near transversals in Abelian groups to analogous objects related to k-weights. We close with Section 5 in which we mention several open questions about k-plexes and k-weights.

A *Latin square* of order n is an $n \times n$ array each of whose cells contains a symbol from a fixed n-set such that no symbol appears twice in any row or column. We can equivalently think of a Latin square L as a set of triples with $(x, y, z) \in L \subset [n]^3$ if and only if the cell in row x and column y contains symbol z. Cayley tables of finite groups and other algebraically interesting loops provide among the most convenient and structured examples of Latin squares. The standard references on Latin squares are due to Dénes and Keedwell [5,6] and Laywine and Mullen [12].

For Latin squares L and L', we say that *L has the block pattern of L'* if L can be represented by a block matrix $[A_{i,j}]_{1 \le i,j \le n}$ where each $A_{i,j}$ is itself a Latin square and blocks $A_{i,j}$ and $A_{i',j'}$ contain the same symbols if and only if $L'(i, j) = L'(i', j')$. In such a case, it follows that each block has the same size, say q, and we say that *L has the q-block pattern of L'*. When we are concerned only with the parity of q, we also refer to the odd-block or even-block pattern. In more classical terminology, a Latin square with the q-block pattern of $(\mathbb{Z}_m, +)$ is said to be of *q-step type* and order qm.

A *k-plex* is a collection of cells of a Latin square that meets each row, column, and symbol precisely k times. The cases $k = 1$ and $k = 2$ correspond to *transversals* and *duplexes*, respectively. The study of transversals dates back at least to Euler [10] and is motivated in part by their intimate connection with mutually orthogonal Latin squares. Wanless provides a helpful historical background on the more recent study of k-plexes for $k \ge 3$ [17].

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.01.007

Generalizing transversals in another direction, a *partial transversal* of length t is a subset of t cells, no two from the same row, column, or symbol, and such a subset is said to be *maximal* if it is not properly contained in another partial transversal. When t is one less than the order of the Latin square, we call the subset a *near transversal*.

Observe that a k-plex of a Latin square L is equivalent to a map from L to the set $\{0, 1\}$ such that the row, column, and symbol sums are all k. From this perspective, it is natural to explore the case when the set $\{0, 1\}$ is replaced by some other set, say $\mathbb{Z}$. It is precisely this case that we consider in this paper. Put another way, the existence of a k-plex is equivalent to the satisfaction of a certain (0, 1)-programming problem, and in this paper we consider the analogous problem over $\mathbb{Z}$.

Suppose $\theta : L \to \mathbb{Z}$ is an integral weight function on the cells of L. For $k \in \mathbb{Z}$, we call θ a *k-weight of L* if its sum over each row, column, and symbol is k. That is, for each index i, we have

$$\sum_{(r,c,i)\in L} \theta(r, c, i) = \sum_{(r,i,s)\in L} \theta(r, i, s) = \sum_{(i,c,s)\in L} \theta(i, c, s) = k.$$

We call θ a *partial k-weight* of L with length t if precisely t row, t column, and t symbol sums are k with each remaining sum being 0. We say that θ *misses* those rows, columns, and symbols whose sums are 0. When t is one less than the order of L, we call θ a *near k-weight* of L. A partial k-weight is said to be *maximal* if, as a vector in $\mathbb{Z}^{n^2}$, it is not dominated by another partial k-weight.

The reader may recognize that partial k-weights are closely analogous to k-homogeneous partial Latin squares.

2. Lemma on partial *k*-weights of Abelian groups

We first recall a lemma of Paige that inevitably comes up in this context.

Lemma 1 (*Paige [15]*). *Suppose $(G, +)$ is a finite Abelian group. If G has a unique involution, then it is equal to $\sum_{g\in G} g$. Otherwise, $\sum_{g\in G} g = 0$.*

The following lemma plays a central role in several of our arguments. It is essentially another version of an argument used by Egan and Wanless to show, among many other things, that certain Latin squares do not contain odd-plexes [9,17]. Our contribution has been to show that the argument applies more generally to partial k-weights.

Lemma 2. *Suppose L is the Cayley table of an Abelian group $(G, +)$ and θ is a partial k-weight whose missing rows R sum to r, missing columns C sum to c, and missing symbols S sum to s. Then*

$$k(s - r - c) = \begin{cases} \sum_{g\in G} g & \text{if } k \text{ is odd and } G \text{ has a unique involution} \\ 0 & \text{otherwise.} \end{cases}$$

Note that when θ is a k-weight (rather than just a partial k-weight), $r = c = s = 0$ and thus the left-hand side is always 0.

Proof. Set $u := \sum_{g\in G} g$. First we consider the sum

$$\sum_{(x,y,z)\in L} \theta(x, y, z)(z - x - y) = \sum_{(x,y,z)\in L} \theta(x, y, z)z - \sum_{(x,y,z)\in L} \theta(x, y, z)x - \sum_{(x,y,z)\in L} \theta(x, y, z)y.$$

We will evaluate the left-hand sum by examining each right-hand sum individually but first note that the result must be 0 since $z - x - y = 0$ for every triple $(x, y, z) \in L$. Grouping the first of the three sums by the z coordinate, we have

$$\begin{aligned} \sum_{(x,y,z)\in L} \theta(x, y, z)z &= \sum_{z\in G} \left(\sum_{(x,y,z)\in L} \theta(x, y, z) \right) z \\ &= \sum_{z\in G\setminus S} kz \\ &= \sum_{z\in G} kz - \sum_{z\in S} kz \\ &= ku - ks. \end{aligned}$$

Likewise, we have

$$\sum_{(x,y,z)\in L} \theta(x, y, z)x = ku - kr \quad \text{and}$$

$$\sum_{(x,y,z)\in L} \theta(x, y, z)y = ku - kc.$$