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Minimum congestion spanning trees in planar graphs

M.I. Ostrovskii *

Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, NY 11439, USA

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ABSTRACT

The main purpose of the paper is to develop an approach to the evaluation or the estimation of the spanning tree congestion of planar graphs. This approach is used to evaluate the spanning tree congestion of triangular grids.

1. Introduction

Let G be a graph and let T be a spanning tree in G (saying this we mean that T is a subgraph of G). We follow the terminology and notation of [5]. For each edge e of T let A_e and B_e be the vertex sets of the components of $T-e$. By $e_G(A_e, B_e)$ we denote the number of edges in G with one end vertex in A_e and the other end vertex in B_e. We define the *edge congestion* of G in T by

$$\mathrm{ec}(G:T) = \max_{e\in E(T)} e_G(A_e, B_e).$$

The number $e_G(A_e, B_e)$ is called the *congestion* in e. The name comes from the following analogy. Imagine that edges of G are roads, and edges of T are those roads which are cleaned from snow after snowstorms. If we assume that each edge in G bears the same amount of traffic, and that after a snowstorm each driver takes the corresponding (unique) detour in T, then $\mathrm{ec}(G:T)$ describes the traffic congestion at the most congested road of T. It is clear that for applications it is interesting to find a spanning tree which minimizes the congestion.

We define *the spanning tree congestion* of G by

$$s(G) = \min\{\mathrm{ec}(G:T) : T \text{ is a spanning tree of } G\}. \tag{1}$$

Each spanning tree T in G satisfying $\mathrm{ec}(G:T) = s(G)$ is called a *minimum congestion spanning tree*. The parameters $\mathrm{ec}(G:T)$ and $s(G)$ were introduced and studied in [13]. This study was continued in [2–4,6–8,10,12,16], where many interesting results were obtained.

The spanning tree congestion is of interest in the study of Banach-space-theoretical properties of Sobolev spaces on graphs; see [14]. Many known results and algorithms related to spanning trees are collected in the monograph [19], but this monograph does not contain any results on the spanning tree congestion. Many related parameters have been introduced in the literature; see [1,9] and references therein. The paper [9] introduced parameters which are more general than the spanning tree congestion.

One of the interesting problems about the spanning tree congestion is to evaluate it for some natural families of graphs. The purpose of this paper is to develop techniques which can be used to evaluate or estimate the spanning tree congestion

* Tel.: +1 718 990 2469; fax: +1 718 990 1650.
E-mail address: ostrovsm@stjohns.edu.

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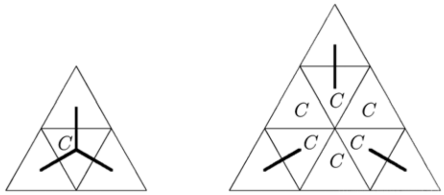

Fig. 1. Examples of center-tail systems.

of planar graphs. The techniques use duality for planar graphs which goes back to Poincarè and Whitney (see [15, Section 8.8.2] and [17,18]) and the notion of a dual tree which is implicitly present in the work of Whitney (see [11, Problems 5.23 and 5.36]). Dual trees were introduced to this area by Hruska [7] who used them to evaluate the spanning tree congestion for rectangular planar grids.

In conclusion we would like to mention that another technique used to estimate the spanning tree congestion is based on the notion of a centroid of a tree (see [19, p. 46] or [13] for the definition) and edge-isoperimetric inequalities. This technique was initiated in [13] and developed in [4] and [10]. It would be interesting to obtain the results for triangular grid (Theorem 2) using isoperimetry.

2. Dual graphs and spanning tree congestion estimates

Let G be a connected plane graph, that is, a planar graph with a fixed drawing in the plane.

Definition 1. The *dual graph* G^* of G is defined as the graph whose vertices are faces of G, including the exterior (unbounded) face, and whose edges are in a bijective correspondence with edges of G. The edge $e^* \in E(G^*)$ corresponding to $e \in E(G)$ joins the faces which are on different sides of the edge e.

Let T be a spanning tree of G. The *dual tree* $T^\sharp$ is defined as a spanning subgraph of G^* whose edge set $E(T^\sharp)$ is determined by the condition: $e^* \in E(T^\sharp)$ if and only if $e \notin E(T)$.

Note. The graph G^* does not have to be a simple graph even when G is simple. It is easy to verify that $T^\sharp$ is a spanning tree in G^* (see [11, Solution of Problem 5.23]). See [5, Section 5.6] and [17,18,11] for information about dual graphs.

Definition 2. Let $e \in E(G)$. We say that e is an *outer edge* if it is an edge which occurs in the boundary of the exterior face and one of the interior faces. For each outer edge e and each bounded face F of G define the *index* $i(F, e)$ as the length of a shortest path in G^* which joins the exterior face O with F and satisfies the additional condition: its first edge is e^*.

Definition 3. A *center-tail* system $\mathcal{S}$ in the dual graph G^* of a plane graph G consists of

(1) A set C of vertices of G^* spanning a connected subgraph of G^*, the set C is called a *center*.
(2) A set of paths in G^* joining some vertices of the center with the exterior face O. Each such path is called a *tail*. The *tip of a tail* is the last vertex of the corresponding path before it reaches the exterior face.
(3) An assignment of *opposite tails* for outer edges of G. This means: For each outer edge e of the graph G one of the tails is assigned to be the *opposite tail* of e, it is denoted $N(e)$ and its tip is denoted by $t(e)$.

See Fig. 1 for examples of center-tail systems.

In the examples shown in Fig. 1 intersections of "thin" line segments are regarded as vertices of the graphs, and there are no other vertices. Edges of the graphs are the corresponding pieces of the "thin" line segments. For the first center-tail system the triangle containing the letter C is the only element of the center. The center of the second center-tail system consists of six faces marked with C. Each of the systems has three tails, shown in Fig. 1 using "fat" lines; we do not show edges joining tips of tails and the exterior face O. The tails going in the upward direction are assigned to be the opposite tails for all outer edges contained in the bottom side of the triangles. Assignments of the opposite tails to edges from other sides of the triangles are made in order to make the assignments rotationally invariant for angles of 120° and 240°. We denote these center-tail systems $\mathcal{S}_3$ and $\mathcal{S}_4$, respectively.

The result below is true for an arbitrary system $\mathcal{S}$ satisfying the relations described above, but to be useful for estimates of the spanning tree congestion, a center should consist of vertices which are far from the exterior face in G^* and opposite tails should be tails which in some natural metric sense go in directions which are opposite to the corresponding edges.

Definition 4. The *congestion indicator* $\mathrm{CI}(\mathcal{S})$ of a center-tail system $\mathcal{S}$ is defined as the minimum of the following three numbers:

(1) $\min_{F,H,f,h}(i(F,f)+i(H,h)+1)$, where the minimum is taken over all pairs F, H of adjacent vertices in the center C and over all pairs f, h of outer edges with $f \neq h$. In the cases where the center contains just one face we assume that this minimum is ∞.