



Hypergraphs induced by algebras of fixed type

Konrad Pióro

Institute of Mathematics, University of Warsaw, ul. Banacha 2, PL-02-097 Warsaw, Poland

ARTICLE INFO

Article history:

Received 12 June 2008

Received in revised form 28 September 2010

Accepted 11 April 2011

Available online 13 May 2011

Keywords:

Undirected and directed hypergraph

Hypergraph orientation

Algebraic representation of hypergraphs

Weak subalgebra lattice

Partial algebra

ABSTRACT

A characterization of the weak subalgebra lattice of a partial algebra of a fixed type is a natural algebraic problem. In Pióro (2000, 2002) [13,15] we have shown that this algebraic problem is equivalent to the following hypergraph question, interesting in itself: When can edges of a hypergraph be directed to form a partial algebra of a fixed type (equivalently, to form a directed hypergraph of a fixed type)? This problem will be solved in the present paper.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

The notions of subalgebra and subalgebra lattice are quite important both in universal algebra and in classical algebras (see e.g., [7,9] or [10]). In the theory of partial algebras the classical notion of subalgebra splits into several completely distinct notions (see e.g., [2] or [5]). On the one hand, these new structures provide additional tools to investigate algebras and varieties of algebras also in the total case. The weak subalgebra lattice together with the subalgebra lattice may yield more information on an algebra than the latter lattice alone. On the other hand, very few properties of weak subalgebras are known. Therefore the starting point of this paper is the following question: When can a given lattice be represented by the weak subalgebra lattice of a partial algebra of a fixed type?

More formally, recall that a partial algebra of type $\langle K, \kappa \rangle$ is a pair $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k \in K} \rangle$, where A is the carrier of $\mathcal{A}$ and for each $k \in K$, $k^{\mathcal{A}}$ is a $\kappa(k)$ -ary partial operation in $\mathcal{A}$, i.e., $k^{\mathcal{A}}$ is defined on a subset of $A^{\kappa(k)}$. Here the type $\langle K, \kappa \rangle$ is a pair, where K is a set of operation symbols and $\kappa: K \rightarrow \mathbb{N}$ is an arity function from K to the set of all the non-negative integers. We say that $\langle K, \kappa \rangle$ is a finite (infinite) type if K is finite (infinite).

Next, a partial algebra $\mathcal{B} = \langle B, (k^{\mathcal{B}})_{k \in K} \rangle$ of type $\langle K, \kappa \rangle$ is a weak subalgebra of a partial algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k \in K} \rangle$ of the same type, if $B \subseteq A$ and $k^{\mathcal{B}} \subseteq k^{\mathcal{A}}$ for all $k \in K$. It is well-known that the set of all the weak subalgebras of $\mathcal{A}$ forms a complete and algebraic lattice $\mathcal{S}_w(\mathcal{A})$ under (weak subalgebra) inclusion $\leq_w$.

The paper [1] gives the following complete characterization of the weak subalgebra lattice.

Theorem 1.1. *A complete lattice $\mathcal{L}$ is isomorphic to the weak subalgebra lattice of some partial algebra if and only if*

- (w.1) $\mathcal{L}$ is algebraic and distributive,
- (w.2) every element is a join of join-irreducible elements,
- (w.3) the set $\text{Ir}(\mathcal{L})$ of all non-zero and non-atomic join-irreducible elements of $\mathcal{L}$ is an antichain with respect to the lattice ordering $\leq_{\mathcal{L}}$,
- (w.4) for each $i \in \text{Ir}(\mathcal{L})$, the set $\text{At}(i)$ of all atoms contained in i is finite and non-empty.

E-mail address: kpioro@mimuw.edu.pl.

Recall (see [6] or [8]) that a non-zero element i of a complete lattice $\mathcal{L}$ is completely join-irreducible if for each set S of elements of $\mathcal{L}$, if i equals the supremum of S (in $\mathcal{L}$), then $i \in S$. An element i is join-irreducible if this condition holds for every two-element set S . A non-zero element a of a lattice $\mathcal{L}$ (with zero) is an atom if there is no non-zero element $b \neq a$ such that $b \leq a$. Recall also (see [7] or [8]) that a complete lattice $\mathcal{L}$ is algebraic if each of its elements is the supremum of a set of compact elements. An element c is compact if for each set S of elements of $\mathcal{L}$, c being contained in the supremum of S implies that c is contained in the supremum of a finite subset of S .

The proof of this theorem is based on the classical result (see [6] p. 83) that an algebraic and distributive lattice $\mathcal{L}$ in which each element is a join of completely join-irreducible elements is isomorphic to a lattice $\mathcal{K}$ of all the order-ideals of the set of all the completely join-irreducible elements of $\mathcal{L}$. This isomorphism maps an element l onto the order-ideal consisting of all the completely join-irreducible elements contained in l . Note that if $\mathcal{L}$ satisfies (w.1)–(w.4) of Theorem 1.1, then all the atoms and all the non-zero and non-atomic join-irreducible elements form the family of all the completely join-irreducible elements of $\mathcal{L}$. We do not use these facts later, but they make it easier to understand why such conditions must be satisfied by a lattice.

Unfortunately, [1] does not contain any information on the type of algebras which can be used to represent a given lattice. This is a very natural algebraic question: When can a given lattice be represented by the weak subalgebra lattice of a partial algebra of a fixed type? In [13,15] we have shown that this algebraic problem is equivalent to some natural hypergraph question, which is interesting in itself. Informally, when can edges of a hypergraph be directed to form a partial algebra of a fixed type (equivalently, to form a directed hypergraph of a fixed type)?

A very particular case of this problem is the following: when can edges of a graph be directed to obtain a functional directed graph, or equivalently, to obtain a partial monounary algebra? Such graphs have been characterized by O. Ore in [11] (see also [3, Chapter 3, Theorem 17]) in the following way: edges of a graph G can be directed to form a functional digraph if and only if each connected component of G contains at most one cycle.

For graphs and unary algebras this problem is solved in [12] (for finite unary types) and in [14] (for infinite unary types). Now we generalize methods from these two papers to solve our problem for arbitrary hypergraphs.

Since we use hypergraphs to represent algebras and lattices, vertex and edge sets may have arbitrary cardinalities, and also multiple edges and isolated vertices are admitted. Moreover, we also use subhypergraphs to represent partial subalgebras. Therefore we will use the following slightly more formal definitions (see e.g., [3,4]).

Definition 1.2. (a) An (undirected) hypergraph H will be represented by a triple $\langle V^H, E^H, I^H \rangle$, where V^H is its vertex set, E^H is its edge set and I^H is its incidence mapping, i.e., I^H is a mapping from E^H to the family of all the finite and non-empty subsets of V^H .

We say that H is finite if its vertex set is finite. Otherwise H is infinite.

- (b) A hypergraph $K = \langle V^K, E^K, I^K \rangle$ is called a weak subhypergraph of a hypergraph $H = \langle V^H, E^H, I^H \rangle$ if $V^K \subseteq V^H$, $E^K \subseteq E^H$ and $I^K(e) = I^H(e) \cap V^K$ for each $e \in E^K$.
- (c) A weak subhypergraph K of a hypergraph H is said to be relative if K contains all the edges of H which have endpoints in K .

Usual subhypergraphs are called weak to stress their relation to weak subalgebras, and also to distinguish them from relative subhypergraphs.

Note that a relative subhypergraph is uniquely determined by its vertex set. Conversely, for each set $W \subseteq V^H$ there is exactly one relative subhypergraph of H with the vertex set W .

We have seen in [13] that the set of all the weak subhypergraphs of a hypergraph H forms a complete and algebraic lattice $\mathcal{S}_w(H) = \langle \mathcal{S}_w(H), \leq_w \rangle$ under (weak subhypergraph) inclusion $\leq_w$. Moreover, it is proved in the same paper that the conditions (w.1)–(w.4) of Theorem 1.1 completely characterize the weak subhypergraph lattice.

By the condition (w.4) of Theorem 1.1 the following definition from [13] is correct.

Definition 1.3. Let $\mathcal{L}$ be a lattice that satisfies (w.1)–(w.4) of Theorem 1.1. Then $\mathbf{U}(\mathcal{L})$ is the hypergraph such that $E^{\mathbf{U}(\mathcal{L})} = \text{At}(\mathcal{L})$, where $\text{At}(\mathcal{L})$ is the set of all the atoms of $\mathcal{L}$, $E^{\mathbf{U}(\mathcal{L})} = \text{Ir}(\mathcal{L})$ and $I^{\mathbf{U}(\mathcal{L})}(e) = \text{At}(e)$ for each $e \in \text{Ir}(\mathcal{L})$.

So defined a hypergraph that is closely connected with $\mathcal{L}$. More precisely, the following result is proved in [13].

Lemma 1.4. Let a lattice $\mathcal{L}$ satisfy (w.1)–(w.4) of Theorem 1.1 and let H be a hypergraph. Then $\mathcal{S}_w(H)$ is isomorphic to $\mathcal{L}$ if and only if the hypergraphs H and $\mathbf{U}(\mathcal{L})$ are isomorphic.

In particular, the lattice of all the weak subhypergraphs of $\mathbf{U}(\mathcal{L})$ is isomorphic to $\mathcal{L}$.

We also need the notion of a directed hypergraph, a simple generalization of a directed graph.

Definition 1.5. A dihypergraph (directed hypergraph) $D = \langle V^D, E^D, I^D \rangle$ is a triple such that V^D is its vertex set, E^D is its edge set and $I^D = \langle I_1^D, I_2^D \rangle$ is its incidence mapping, i.e., I_1^D is a mapping from E^D to the family of all the finite (possibly empty) subsets of V^D and I_2^D is a mapping from E^D to V^D .

For a directed edge e of D , $I_1^D(e)$ will be called the initial set of e and $I_2^D(e)$ the final vertex of e .

Download English Version:

<https://daneshyari.com/en/article/4648644>

Download Persian Version:

<https://daneshyari.com/article/4648644>

[Daneshyari.com](https://daneshyari.com)