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Group divisible designs with three unequal groups and larger first index

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ABSTRACT

We show the necessary conditions are sufficient for the existence of group divisible designs (or PBIBDs) with block size $k = 3$ with three groups of size $(n, 2, 1)$ for any $n \geq 2$ and any two indices with $\lambda_1 > \lambda_2$.

1. Introduction

A *group divisible design*, *or* GDD, is a collection of k-subsets (called blocks) of a set V with v elements, where the set V is partitioned into g groups of sizes $v_1, v_2, \ldots, v_g$. Each pair of elements from the same group occurs in exactly λ_1 blocks; and each pair of elements from different groups occurs in exactly λ_2 blocks. Pairs of symbols occurring in the same group are known to statisticians as *first associates*, and pairs occurring in different groups are called *second associates*. Of course, if the indices λ_1 and λ_2 were equal, then the design would be a BIBD [3,4], and we avoid this possibility throughout, requiring in Section 3 in fact that λ_1 be greater than λ_2.

It is useful to describe GDDs graphically. Let λK_n denote the graph on n vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. If $\lambda = 1$ then we simply write $G_1 \vee G_2$. A G-decomposition of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Hence a GDD$(v = m+n, 2, 3, \lambda_1, \lambda_2)$ is equivalent to a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. In this graph theoretic setting, edges joining vertices (symbols) in the same group are referred to as pure edges, whereas edges joining vertices in different groups are called mixed edges. In general, if the number of groups is less than the block size, or of unequal size, then the construction of such GDDs is considered more difficult. All GDDs studied in this paper will have $k = 3$, and we abbreviate the notation using a group size vector and an index vector: for instance GDD$(n_1, n_2, n_3; \lambda_1, \lambda_2)$.

The designs in this note historically were called group divisible designs [1], or GDDs, but are called partially balanced incomplete block designs (PBIBDs) of group divisible type in [3], reserving GDD strictly for the $\lambda_1 = 0$ case. In [9] they are called group association designs but we use the older name. Complete results for groups of equal size (for $k = 3$) appear in [6,7].

GDDs with two association classes, with $k = 3$, GDD$(m, n; \lambda_1, \lambda_2)$, in which each group intersected each block, were investigated in [5]. In [8] the present authors investigated GDDs with two groups of equal size with $k = 4$. In [2], necessary

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and sufficient conditions were found for GDD$(1, n; 1, \lambda)$, and GDD$(n, 2, 1; \lambda_1, \lambda_2)$ for $n \in \{2, \dots, 6\}$ for $k = 3$. In [10], necessary and sufficient conditions were found for GDD$(1, 1, n; 1, \lambda)$. In [9], the necessary and sufficient conditions are given for GDD$(n, 1, 1; \lambda, 1)$, and both GDD$(1, 1, 1, n; 1, \lambda)$, and GDD$(1, 1, 1, n; \lambda, 1)$.

In this paper, we deal with the $k = 3$ case for three groups with sizes $(n, 2, 1)$, an investigation which continues in some fashion each of the last three papers cited. This exact case was considered in [2] which showed the necessary conditions were sufficient for $n = 2, 3, 4, 5, 6$ and any indices. In this paper, we completely solve the $(n, 2, 1)$-case with $\lambda_1 > \lambda_2$ for any n. In Section 2, we review background information and summarize what is needed from earlier work so that this paper may be read independently. The new results are in Section 3.

The following notation for sets of triples will be used throughout the paper for our constructions.

(1) Let $T = \{x, y, z\}$ be a triple and $a \notin T$. We use $a * T$ for the three triples $\{a, x, y\}$, $\{a, x, z\}$, $\{a, y, z\}$. If $\mathcal{T}$ is a set of triples, then $a * \mathcal{T}$ is defined as $\{a * T : T \in \mathcal{T}\}$.
(2) Let $e = uv$ be an edge of a graph G. We use $a + e$ for the triple $\{a, u, v\}$. If X is a set of edges of a graph G, then $a + X$ is defined as $\{a + e : e \in X\}$.
(3) By $\{a, b, c\} \times j$ we mean use j copies of the block $\{a, b, c\}$.
(4) If A is a set of points (vertices) with $|A| = v$, we use the notation BIBD$(A, 3, \lambda)$ to mean a BIBD$(v, 3, \lambda)$ on the points of A.

2. GDDs with three groups of unequal size

In this section, we give necessary conditions for the existence of GDDs with three groups of unequal size. The three groups will be $G_1 = \{1, 2, \dots, n\}$, $G_2 = \{a, b\}$, and $G_3 = \{z\}$ with sizes, respectively of n, 2, and 1. We begin with an infinite family of examples.

Example 1. Let $n = 3t$. We give a family of GDD$(n, 2, 1; 2n+2, 2)$, where $G_1 = \{1, 2, \dots, n\}$, $G_2 = \{a, b\}$ and $G_3 = \{z\}$ are the three groups. We suppose there exists a BIBD$(n, 3, \mu)$ which has (at least) one parallel class C. Then use the following blocks for the GDD. Use $z * C$, that is, for each block $\{c, d, e\}$ in C, form the three blocks $z * \{c, d, e\}$. In this way point z meets each point of G_n twice. Use two copies of block $\{a, b, j\}$ for each $j \in G_1$ and two copies of block $\{a, b, z\}$. It follows that $\lambda_2 = 2$. Points a, b of G_2 already meet in $2n + 2$ blocks, and so we require $\mu = 2n + 2$. It follows that $\lambda_1 = 2n + 2$. The parameter n may be taken to be $6s+3$ for $s \geq 0$ or $6s$ for $s \geq 1$, since resolvable BIBDs are known to exist for $\lambda = 2$ and such n [see Section 7.4 of [3]; if $n = 6$, a resolvable BIBD$(6, 3, 4)$ exists]. It is especially noteworthy that, if $n = 3u$, then u and λ_1 may increase arbitrarily while the second index stays fixed at 2. This may be contrasted with those results in [2] where n is small and $\lambda_2 > \lambda_1$.

2.1. *Necessary conditions for the three group case*

Necessary conditions on the existence of a GDD$(n_1, n_2, n_3, \lambda_1, \lambda_2)$ can be obtained from a graph theoretic point of view. The existence of a GDD$(n_1, n_2, n_3; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $(\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2}) \vee_{\lambda_2} \lambda_1 K_{n_3}$, from here on designated simply as $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ by associativity of joins and folds. The graph $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ is of order $n_1 + n_2 + n_3$ and size $\lambda_1 \left[\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}\right] + \lambda_2(n_1 n_2 + n_2 n_3 + n_3 n_1)$. It contains n_1 vertices of degree $\lambda_1(n_1 - 1) + \lambda_2(n_2 + n_3)$, n_2 vertices of degree $\lambda_1(n_2 - 1) + \lambda_2(n_1 + n_3)$, and n_3 vertices of degree $\lambda_1(n_3 - 1) + \lambda_2(n_1 + n_2)$. Thus the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ implies:

Lemma 1. *For a GDD$(n_1, n_2, n_3;\ \lambda_1, \lambda_2)$, with β blocks, it is necessary that:*

(1) $3 \mid \lambda_1 \left[\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}\right] + \lambda_2(n_1 n_2 + n_2 n_3 + n_3 n_1)$,
(2) $2 \mid \lambda_1(n_1 - 1) + \lambda_2(n_2 + n_3)$, $2 \mid \lambda_1(n_2 - 1) + \lambda_2(n_1 + n_3)$ *and* $2 \mid \lambda_1(n_3 - 1) + \lambda_2(n_1 + n_2)$, *and*
(3) $\beta = \frac{1}{6}\left(\lambda_1(n_1^2 + n_2^2 + n_3^2 - n_1 - n_2 - n_3) + 2\lambda_2(n_1 n_2 + n_1 n_3 + n_2 n_3)\right)$.

2.2. GDD$(n, 2, 1; \lambda_1, \lambda_2)$

Now we continue to investigate all triples of integers $(\lambda_1, n, \lambda_2)$ in which a GDD$(n, 2, 1; \lambda_1, \lambda_2)$ exists, where $\lambda_i \geq 1$. First, we specialize the formulas of the previous section to our situation: $n_1 = n$, $n_2 = 2$ and $n_3 = 1$, involving the sets $G_1 = \{1, 2, \dots, n\}$, $G_2 = \{a, b\}$, and $G_3 = \{z\}$ respectively. After some simplification, we obtain

(1) $\lambda_1 (n(n-1) + 2) + \lambda_2 \equiv 0 \pmod 3$,
(2) $\lambda_1(n - 1) + \lambda_2 \equiv 0 \pmod 2$, $\lambda_1 + \lambda_2(n + 1) \equiv 0 \pmod 2$, and $\lambda_2 n \equiv 0 \pmod 2$, and
(3) $\beta = \frac{1}{6}\left(\lambda_1(n^2 - n + 2) + 2\lambda_2(3n + 2)\right)$.

It is convenient in what follows to have available the replication numbers r_1, r_2, and r_3, for their respective groups. These are $r_1 = [\lambda_1(n - 1) + 3\lambda_2]/2$, $r_2 = [\lambda_1 + \lambda_2(n + 1)]/2$, and $r_3 = (n + 2)\lambda_2/2$.